Exploring the Damage Number of a Graph

BY

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Abstract

Cops and Robbers is a well-studied pursuit-evasion game, where a set of cops aims to capture a robber on a graph while the robber attempts to avoid capture indefinitely. We consider the damage number of a graph, in which the original goal of the players in Cops and Robbers is altered. In this game, the robber's goal is to visit the maximum number of unique vertices in a graph while the cops' goal is to minimize this number. Although the damage number of a graph can be studied with any number of cops, we will consider the game with a single cop and robber. We present new results which characterize graphs with damage number one, provide an improved lower bound on the damage number for any graph, and determine the exact value for the damage number of graph joins. Following this, we provide a relationship between cut-sets and the damage number by defining a new graph property, star cut-sets, and explore this property through cactus graphs.

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To my community, both family and friends, thank you for the unconditional care throughout my studies. Individual achievements serve as markers for us to reflect on, and value, those who carry us along the way. I hope to signify to each of you, individually, the impact that you have made in my life and strive to return the favour.

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Chapter 1

Introduction

Cops and Robbers, first introduced by Quilliot [14] and independently considered by Nowakowski and Winkler [13], is a game played graphs where a set of cops attempt to capture a robber. The cops and robber are restricted to vertices of a reflexive graph and travel along edges between their current vertex and any neighbouring one. The introduction of this game has led to an extensive research field, investigating both aspects of the original game and new game variants. A foundational question in research asks how many cops are required to guarantee the capture of the robber on a given graph.

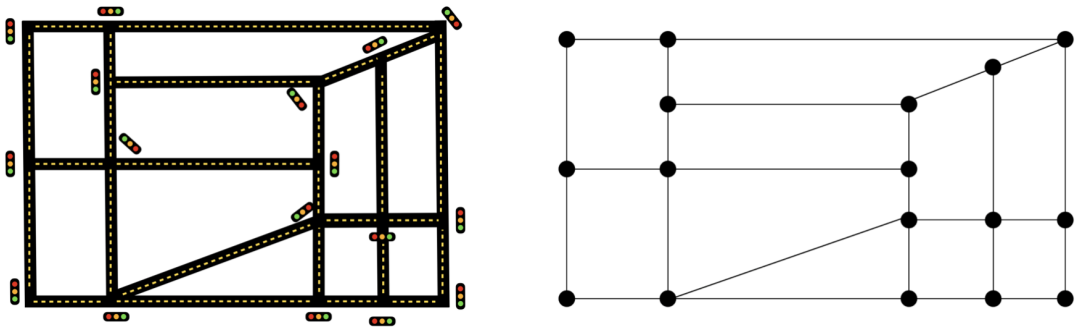


Figure 1.1: A city map (left) and its corresponding graph (right).

The game can be used to model realistic pursuit-evasion situations, where an individual is avoiding being caught by a police officer. The most obvious implementation comes from recasting a city or map as a graph. Here, an edge represents a road while a vertex represents an intersection between two or more roads. Figure 1.1 provides

an example of how a map can be modeled using a graph.

Strategies in Cops and Robbers may also be used when when playing video games such as Pac-Man. For those who are not familiar with this cult classic, Pac-Man is trapped inside of a maze with non-stationary obstacles. A player's goal is to move Pac-Man around the maze and eat 'dots' while avoiding being touched by various coloured ghosts. Although Cops and Robbers does not capture the dot eating aspect of the game, it does capture the overarching goal of the player: to avoid being touched by the ghosts. Here, we can think of the ghosts as cops, the robber as Pac-Man, and the maze as the graph. In Section 1.2, we will formalize the ruleset of Cops and Robbers and describe assumptions we make when playing the game through examples.

1.1 Preliminary Definitions

Before we formalize the game of Cops and Robbers, we will begin by introducing some common graph theoretic notation. The following definitions and their associated notation notation are consistent with [9]. For any graph G, we write $G = (V(G), E(G))$ where $V(G)$ is the vertex set and $E(G)$ is the edge set of a graph G. The *order* of a graph G is the cardinality of the vertex set, $|V(G)|$. We say that a graph is *connected* if there exists a path between any two vertices. Any graph that is not connected is *disconnected*. Graphs G and H are *isomorphic* graphs if there exists a bijective mapping $f : V(G) \to V(H)$ such that $vw \in E(G)$ if and only if $f(v)f(w) \in E(H)$. If G and H are isomorphic then we denote this relationship as $G \cong H$. A graph G is *reflexive* if there exists an incident loop on every vertex in G. In this thesis, we will only consider non-empty graphs that are both reflexive and connected, unless otherwise indicated. In examining further graph properties, we will explore definitions on graph J, shown in Figure 1.2. Note that although Figure 1.2 does not show incident loops on every vertex of J, we assume they exist. In fact, we assume incident loops exist in every illustration of a graph.

Two vertices u and v are *adjacent* if there exists an edge, uv, in $E(G)$. We say $u \sim v$ if u and v are adjacent and $u \not\sim v$ otherwise. The *open neighbourhood* of a vertex $v \in V(G)$ is the set of vertices that v is adjacent to, and is denoted $N(v)$. The *closed neighbourhood* of v is defined as $N(v) \cup \{v\}$ and is denoted $N[v]$. We say that u is a *neighbour* of v if $u \in N(v)$. The *degree* of a vertex v is the cardinal $|N(v)|$ and is denoted $deg(v)$. The *minimum degree* of a graph is $\delta(G)$ with its analogous counterpart being the *maximum degree*, $\Delta(G)$. A *dominating set* is a set of vertices

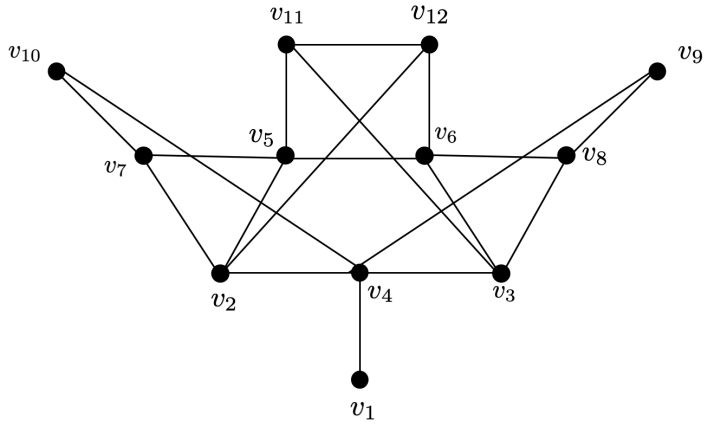


Figure 1.2: A graph J.

$\{v_1, v_2, \ldots, v_k\}$ such that $N[v_1] \cup N[v_2] \cup \ldots \cup N[v_k] = V(G)$. We denote $\gamma(G)$ as the number of vertices in the smallest dominating set in G. An *independent set* is a set of vertices in a graph G in which no vertices are adjacent. The *girth* of a graph is the length of the shortest cycle in the graph.

For example, consider the vertex v_3 in the graph J (see Figure 1.2). We know that v_3 is adjacent to vertex v_4, v_6, v_8, and v_{11}. Therefore $N(v_3) = \{v_4, v_6, v_8, v_{11}\}$, $N[v_3] = \{v_3, v_4, v_6, v_8, v_{11}\}$ and $deg(v_3) = 4$. In J, the minimum and maximum degree are $\delta(J) = 1$ and $\Delta(J) = 5$, where $deg(v_1) = 1$ and $deg(v_3) = 4$. A minimum dominating set in J is $\{v_4, v_5, v_6\}$, so $\gamma(J) = 3$. The set $\{v_9, v_{10}\}$ forms an independent set because $v_9v_{10} \notin E(J)$. Finally, the girth of this graph is three where vertices v_7, v_2 and v_5 form the smallest cycle in the graph.

We say that the *induced subgraph* of a graph G induced by a subset $S \subseteq V(G)$, denoted $G[S]$, is the graph formed by $G[S] = (S, \{vw|\ v, w \in S, vw \in E(G)\})$. A vertex v is a *cut-vertex* if $G[V(G)\backslash\{v\}]$ results in more than one connected component. For a connected graph G, a set $S \subseteq V(G)$ is a *cut-set* if $G[V(G) \setminus S]$ results in more than one connected component. Furthermore, a *clique* in a graph is a set $S \subseteq V(G)$ such that $G[S]$ is isomorphic to the complete graph K_n, for some $n \in \mathbb{N}$.

Again returning to the graph J, we can see that v_4 is a cut-vertex since $J[V(J)\backslash v_4]$ results in v_1 being an isolated vertex and the rest of the graph forming a connected component. The set $\{v_2, v_3, v_4, v_5, v_6\}$ forms a cut-set since it partitions the graph into four components with vertices $\{v_1\}$, $\{v_7, v_{10}\}$, $\{v_{11}, v_{12}\}$, and $\{v_8, v_9\}$ respectively. Furthermore the vertices $\{v_2, v_5, v_7\}$ form a clique since $J[\{v_2, v_5, v_7\}] \cong K_3$.

We now provide additional definitions that are consistent with those in [15]. The

distance from vertex u to v, denoted $d(u, v)$, is the length of the shortest path from u to v. The *eccentricity* of a vertex v is the maximum distance from v to any other vertex in the graph. The *radius* of a graph, denoted $rad(G)$, is the minimum eccentricity of all vertices in G. The *center of a graph* is any vertex that has an eccentricity equal to the radius of a graph.

In J, we know that the radius of the graph is three since the eccentricity of v_5 is three and is the smallest eccentricity of any vertex in J. Therefore v_5 is the center of the graph since its eccentricity is equal to the radius.

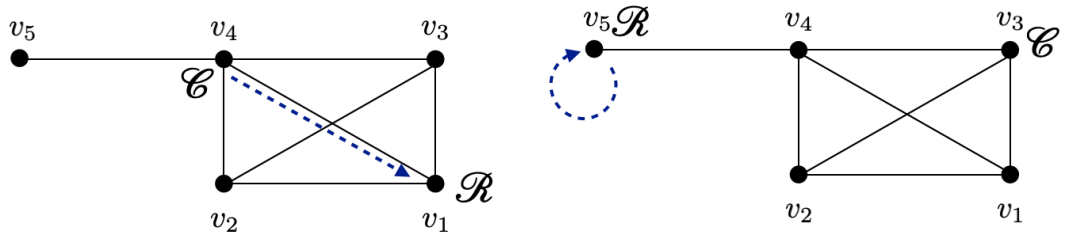
Before concluding this section, we will define graph classes that will be considered later in the thesis. *Trees* are graphs which contain no cycles. A *multi-partite* graph is a graph that can be partitioned into multiple independent sets. A multi-partite graph is *complete* if there is an edge between every pair of vertices that belong to different partition sets. A *chord* is an edge joining two vertices that are not adjacent in a cycle. We say a graph is *chordal* if each of its cycles of four or more vertices has a chord. A graph G is k-regular if for all $v \in V(G)$, $deg(v) = k$ for some $k \in \mathbb{N}$.

Other helpful background information and definitions will be introduced as they become relevant in the discussion.

1.2 Cops and Robbers: Formally

The game of *Cops and Robbers* (also sometimes called *Cops and Robber*) is a two-player game played on a reflexive graph. One player controls a finite set of cops, while the other controls a single robber. Initially, the cops choose a multiset of vertices to occupy. Following this, the robber chooses a vertex to occupy in the graph. At each time-step, a subset of cops can move to any vertex adjacent to their vertex set, the robber then moves to a vertex in the neighbourhood of their current vertex. Therefore, at the beginning of any time-step, the cops move first followed by the robber. Since we are playing on reflexive graphs, either player can *pass* during their move, which equates to traversing the incident loop and remaining on their current vertex. We assume both players are playing with perfect information, meaning players know each other's positions and the layout of the graph at all times.

If, after a finite number of moves, at least one cop occupies the same vertex as the robber, then the robber is *captured*. The cops' objective is to capture the robber, while the robber aims to avoid capture indefinitely. As a result, we assume that each player plays *optimally*, where at every time step, each cop and the robber move to



(a) The optimal strategy of the cop, given the robber's positioning.

(b) The optimal strategy for the robber, given the cop's positioning.

Figure 1.3: A graph G where G isomorphic to K_4 with an appended leaf.

a vertex which is most helpful in achieving their respective goals. To illustrate this assumption, consider the complete graph K_4 with a leaf appended (see Figure 1.3). Here, $\mathscr{C}$ represents the cops, $\mathscr{R}$ represents the robber, and a dotted line indicates a player's movement on their turn. We will be consistent with this labelling of the cop and robber in future figures. Suppose that it is the cop's turn and there is a cop on v_4 and a robber on v_1, as shown in Figure 1.3(a). The cop has many options for where they can move; they can move to any vertex in $N[v_4]$ and $|N[v_4]| = 5$. However, the cop's best option is to move to vertex v_1 and capture the robber, since it is the cop's objective to capture the robber. Therefore, in the analysis of the game, we would assume the cop executes this move. The same idea can be applied to the robber. Suppose instead that it's the robber's turn and they are located on v_5, and additionally, the cop is on v_3 (see Figure 1.3(b)). Note that $v_5 \notin N[v_3]$, but for all $v_i \in N(v_5)$, $v_i \in N[v_3]$. Therefore, the robber is occupying the only vertex which allows them to avoid immediate capture. As a result, on their turn, the robber will pass and remain on vertex v_5.

For the graph in Figure 1.3, one cop can be successful in capturing the robber: place the cop on vertex v_4. Since $N[v_4] = V(G)$, the cop can move to the robber's location on their turn and capture the robber. This is not always the case. Consider the cycle graph of order eight, denoted C_8 and shown in Figure 1.4. If we were to play this game with one cop, the robber could avoid capture forever. Assume, without loss of generality, that the cop places themselves on a vertex v_1 and the robber places themselves on vertex v_3. Since the robber is not adjacent to the cop, we know the robber will not be captured on the first round. In fact, the robber can avoid capture indefinitely by mimicking the movements of the cop. If the cop passes then so does the robber and if the cop moves either clockwise or counterclockwise, then so does

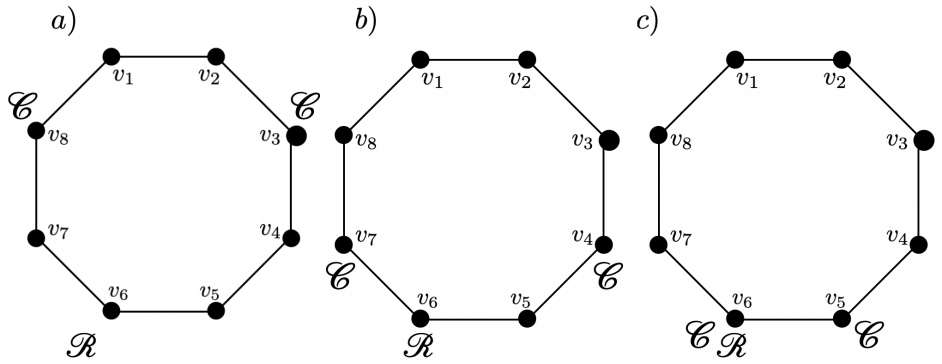


Figure 1.4: An illustration of how two cops capture a robber in C_8.

the robber. Using this strategy, the robber can maintain a distance of two away from the cop to avoid capture. As a result, we need two cops to guarantee the capture of the robber, as illustrated in Figure 1.4. If we place the cops on v_3 and v_8, then the robber is forced to place themselves on v_5 or v_6, as they are the only vertices not in $N[v_3] \cup N[v_8]$. Without loss of generality, suppose the robber places themselves on v_6. On the first round, the cops can move to v_4 and v_7 respectively. In doing so, to avoid capture, the robber can only move to vertex v_5 or remain on vertex v_6 on their turn. Part *b*) in Figure 1.4 shows the positioning of the robber on the first round, if they choose to remain on vertex v_6. In the second round, the cops can move to vertex v_5 and v_6 to ensure the capture of the robber, as illustrated in part *c*). Note that if the robber moved to vertex v_5 instead of remaining on v_6, the cops would still be able to capture the robber by moving to vertices v_5 and v_6 on round two. Therefore, we can guarantee, no matter the strategy the robber executes on C_8, two cops are able to capture the robber.

Cops and Robbers focuses on finding the minimum number of cops required to capture the robber on a graph. We could have placed another cop on vertex v_6 initially, so the cops occupy dominating set in the graph. When cops occupy a a dominating set in a graph, the robber would be immediately captured in the cops' first move since every vertex in the graph is a distance of one away from at least one of the cops. Although three cops can guarantee capture of the robber, since the same goal can be achieved with a smaller number of cops, placing cops on every vertex of C_8 is not the strategy executed by the cops. The minimum number of cops is known as the *cop number of a graph*, denoted $c(G)$. When $c(G) = 1$, we say that G is a *cop-win graph*. If $c(G) = k$ and $k > 1$, then we say the graph G is *k-cop win*. Finding

the cop number of a graph is the main focus for those who study this game.

Placing a cop on every vertex would result in the robber being trivially captured, therefore the cop number is well-defined and $1 \leq c(G) \leq |V(G)|$. These bounds often serve little use in finding the minimum number of cops required to capture the robber. For example, on a cycle graph where $|V(C_n)| = n$, as n increases so does the difference between $|V(C_n)|$ and $c(C_n)$. Therefore, we can generalize the strategy shown in Figure 1.4 for any cycle C_n where n is the order of the cycle and $n \geq 4$, making $c(C_n) = 2$ [4].

1.3 Pitfalls and Cop-Win Graphs

Significant progress has been made in determining the cop number on specific graphs. We will now take a closer look at key results in the field, starting with a complete characterization of cop-win graphs.

Recall, in Cops and Robbers, we assume both players act optimally. When a robber is captured by a cop in a graph, a cop must move along an edge to the neighbouring vertex where the robber is located. This means that for the robber to be caught, they must be in a situation where every vertex they can move to can also be accessed by a subset of cops. When there is only one cop on the graph, this situation occurs when the robber's neighbourhood is contained in the neighbourhood of the cop. More precisely, the robber is located on some vertex $v \in V(G)$ such that $N[v] \subseteq N[u]$ where u is the position of the cop. In such cases, we say that vertex v is a *pitfall* (sometimes also called a corner, trap, or irreducible) with vertex u being its *attack* vertex. To explain further, we will refer back to graph G in Figure 1.3, which we used to examine optimal strategies. The vertex v_5 is a pitfall with its attack vertex being v_4, since $N[v_5] = \{v_4, v_5\}$ and $N[v_4] = \{v_1, v_2, v_3, v_4, v_5\}$, so $N[v_5] \subseteq N[v_4]$. There are other pitfalls in G as well; to name a few, v_1 and v_3 are pitfall vertices with their attack vertices being v_4 and v_2 respectively. Pitfall vertices contain the requirements for a robber to be caught by one cop in the last round, which brings us to an important result in the literature.

Lemma 1.1 ([13])**.** *If G is a cop-win graph, then G has at least one pitfall.*

The proof for Lemma 1.1 is quite simple. Consider the second last move of the cop where v is the positioning of the robber and u is the positioning of the cop. Assuming G is a cop-win graph, if the robber passes during their turn, the cop must catch them,

therefore $v \in N[u]$. If the robber moves to a neighbouring vertex, again the cop must also be able to move to this vertex, therefore $N(v) \subseteq N(u)$. Putting these conditions together we have that $N[v] \subseteq N[u]$.

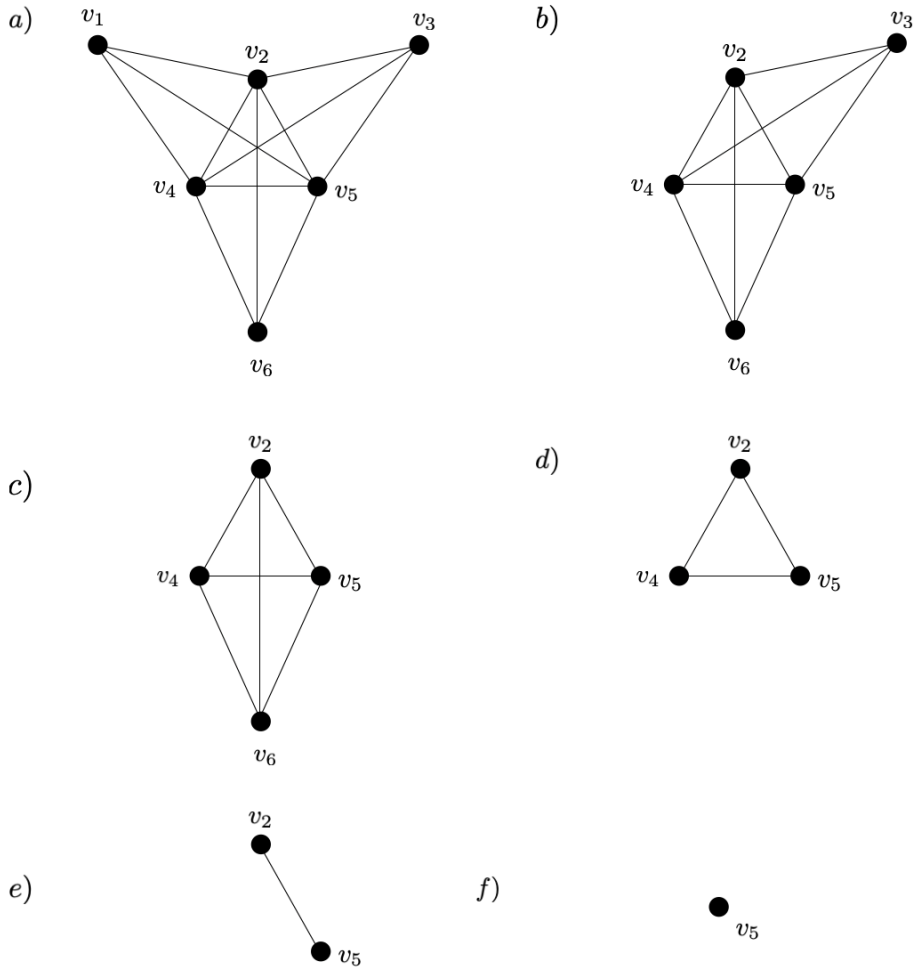


Figure 1.5: A graph G being dismantled.

We will now introduce Theorem 1.2, which uses pitfalls to produce a stronger result than Lemma 1.1. A graph is *dismantlable*, if successively deleting pitfalls in the graph results in a single vertex. Every tree is dismantlable because we can delete leaves repeatedly until only one vertex remains. Figure 1.5 shows a graph G that is dismantlable by deleting vertices in the following order: $(v_1, v_3, v_6, v_4, v_2)$. Note v_1, v_3 and v_6 are pitfalls in $V(G)$ with v_2 being their attacking vertex. Therefore, deleting these vertices in successive order results in a graph isomorphic to K_3. At this point, v_5 is an attack vertex in $G[V(G) \setminus \{v_1, v_3, v_6\}]$ and v_4 and v_2 are pitfalls. Deleting

v_4 and v_2, leaves us with the single vertex v_5. This is just one way to dismantle the graph, for example, a single vertex would also be obtained by deleting v_3, v_2, v_1, v_6, and then v_4 in successive order.

Theorem 1.2 ([13])**.** *G is a cop-win graph if and only if G is dismantlable.*

The proof for Theorem 1.2 uses mathematical induction. For the forward direction, the base case is an isolated vertex, which is trivially dismantlable. For a graph with a larger order, let v be pitfall vertex. Using the inductive hypothesis, we can see that the cop can capture the robber on the graph formed by removing v. Furthermore, the cop can use the same strategy on the original graph: pretend the robber is on the attack vertex of v whenever the robber is actually on v. Using this principle, the robber will either be caught by the cop or they are on v and the cop is on the attack vertex of v. From here the cop can win in one more move, since v is a pitfall vertex. Considering the backwards direction, from Lemma 1.1, we know that every cop-win graph has at least one pitfall vertex. Assume v is a pitfall in a graph H, removing v must result in another cop-win graph. If not, the robber could restrict themselves to $V(H) \setminus \{v\}$ in H and avoid capture indefinitely.

It follows by induction that every finite cop-win graph is dismantlable, which provides a complete characterization for cop-win graphs. Using Theorem 1.2, we can confirm that C_8 is not cop-win because it does not contain any pitfalls.

1.4 Multi-Cop Strategies and Game Extensions

For a graph G that does not contain any pitfall vertices, by Lemma 1.1, we know $2 \leq c(G) \leq |V(G)|$. To tighten the bound on the cop number, need to examine strategies of the cops and robber further. Unless otherwise indicated, information in this section is summarized from [4].

Although Cops and Robbers was first introduced in the early 1980s, the characterization of k-cop win graphs remains unsolved. Despite this, we know that such a characterization exists. In [7], they give both a relational and elimination-ordering for such graphs. Since a complete characterization has not been found, we can provide upper and lower bounds for the cop number of k-cop win graphs by looking at specific graph properties. For an example of how we can use graph properties to determine the cop number of a graph, consider the graph H featured in Figure 1.6.

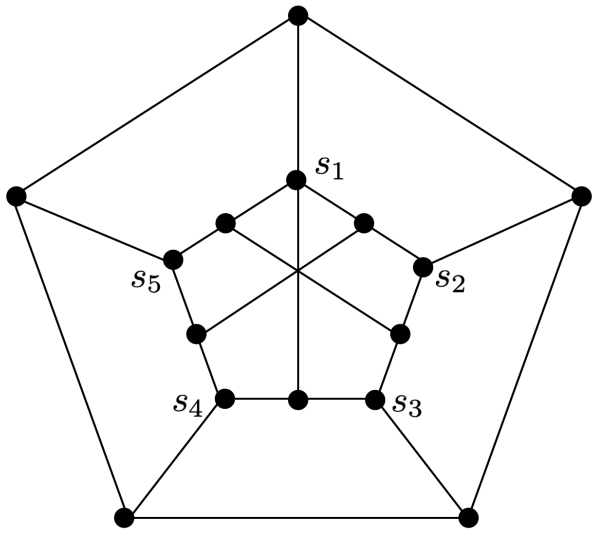


Figure 1.6: A graph H with *girth* of 5.

Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of vertices labeled in Figure 1.6. The set S forms a dominating set of the graph H. By placing the cops on this dominating set, the robber is forced to vertices in $V(H) \setminus S$. However, since S forms a dominating set, no matter where the robber places themselves on $V(H) \setminus S$, the cops can move to capture the robber on their first turn. This being said, the explanation given does not guarantee that this is the minimum number of cops required the capture the robber; therefore, we can only conclude $c(G) \leq 5$. Upper bounds require a specific focus on the cops' strategy, while considering the robber's position more generally. In the above example, we were not concerned with the starting position of the robber since every vertex resulted in capture. To provide an upper bound, we must identify a strategy for the cops to guarantee capture over all of the possible strategies of the robber. In contrast, lower bounds consider a strategy of the robber and show that they can avoid capture over all of the possible cop strategies. Aigner and Fromme [1] provided an early result on a lower bound for graphs with *girth* of at least five. We now present their result.

Theorem 1.3 ([1])**.** *If G is a graph with girth of at least 5, then $\delta(G) \leq c(G)$.*

This proof uses contradiction by assuming $\delta(G) - 1$ cops are playing the game and showing that the robber can avoid capture indefinitely. Aigner and Fromme show the robber can survive the first round of the game because there is a vertex in G not

in the union of closed neighbourhood sets of each of the $\delta(G) - 1$ cops' positioning. Assume that the robber places themselves on such a vertex.

Consider round $t \geq 0$ and assume the robber occupies u_t with the following property, where C is the set of vertices occupied by the $\delta(G) - 1$ cops.

$$(C_t) : \text{The vertex } u_t \text{ is not adjacent to any vertex in } C.$$

We know such a vertex exists, as the robber placed themselves on a vertex with this property at the start of the game. From this definition we can conclude that the robber cannot be captured in round t. If (C_t) holds for all $t \geq 0$ and u_t exists in G then the robber could keep moving to u_t. We proceed by induction on t. The base case is already stated, as u_t exists when $t = 0$. Suppose this is true for $t - 1$, meaning the robber is on a vertex u_{t-1} that is not adjacent to a cop. Since the girth of the graph is at least five, each cop is only adjacent to at most one neighbour of u_{t-1}. However since u_{t-1} has a degree of $\delta(G)$ or more, the robber can move to u_t not adjacent to any cop.

Again, considering H in Figure 1.6, $\delta(H) = 3$, therefore $3 \leq c(G) \leq 5$. By considering the strategy of the robber, we can find a tighter lower bound on the cop number. Notice that in the explanation of Theorem 1.3, we considered both the strategy of the robber and the cops. This differs from the explanation of the upper bound, which only considered the strategy of the cops.

In Section 1.2 we described how the parameter of interest is the minimum number of cops required to guarantee capture of the robber. This does not necessarily need to be the focus of our studies. The field of Cops and Robbers has been expanded to study different parameters using a similar ruleset or game framework. For example, in various pursuit-evasion scenarios, it is useful to know the minimum time it takes for a single cop (or set of cops) to capture the robber. Bonato et al. in [2] introduce and formalize this new graph parameter.

Definition 1.4. *Let G be a k-cop win graph where the robber is caught by k cops in the t-th round. We say that the k **capture time** of G is t and write $capt_k(G) = t$.*

For cases when $k = 1$, we write $capt(G)$. As stated in [2], the capture time of a graph may be viewed as the temporal counterpart of the cop number and is equivalent to the length of a game. Bonato et al. provide some bounds on the capture time for cop-win graphs.

Theorem 1.5 ([2]). *If G is a cop-win graph then $capt(G) \leq |V(G)| - 3$, provided $|V(G)| \geq 5$.*

The proof proceeds by induction on the order of a cop-win graph. Using a direct check of all cop-win graphs where $|V(G)| = 5$ we could see that $capt(G) \leq 2$, giving us the base case. For the inductive hypothesis, assume $capt(G) \leq |V(G)| - 3$ for all cop-win graphs where $|V(G)| \leq n$ and $n \geq 5$. Consider a cop-win graph G with $n+1$ vertices. Since G is cop-win it must contain a pitfall, say $u \in V(G)$. Therefore the graph $G \setminus \{u\}$ will have the same capture time. We will return to capture time in Section 1.5.3.

1.5 The Damage Number of a Graph

1.5.1 Introduction and Motivation

In [8], Cox and Sanaei introduce a version of the game where the robber damages every distinct vertex they visit. The minimum number of vertices the robber can damage is called the *damage number* of a graph and is denoted by $dmg(G)$. Motivation for the authors' introduction of this parameter is guided by the following questions. What happens in pursuit-evasion scenarios where we do not have access to, or enough resources for, the minimum number of cops it takes to capture the robber? Although the robber can stay on the graph indefinitely, how might the cops minimize the harm the robber can do on a graph? What if the damage inflicted by the robber is particularly costly or severe?

We imagine that the robber damages a vertex in-between round i and $i+1$ and that any vertex damaged in-between rounds remains damaged permanently after round $i+1$. Both players can still occupy any vertex in the graph, even if they are damaged. In this version of the game, we also assume both players play optimally, however the goals of each player differ from the original game. The robber always moves to maximize the number of distinct vertices they can damage while the cops move to minimize the number of distinct vertices the robber can damage. Figure 1.7 shows the optimal strategy of both players on the graph P_7. In a) the cop and robber choose their starting position. In round one, the cop and robber both move right on the path, damaging one vertex in the process as shown in b). Again, in c), the cop moves to the right while the robber remains on their current vertex, damaging it in the process. Finally, as d) shows, the cop captures the robber in round three without

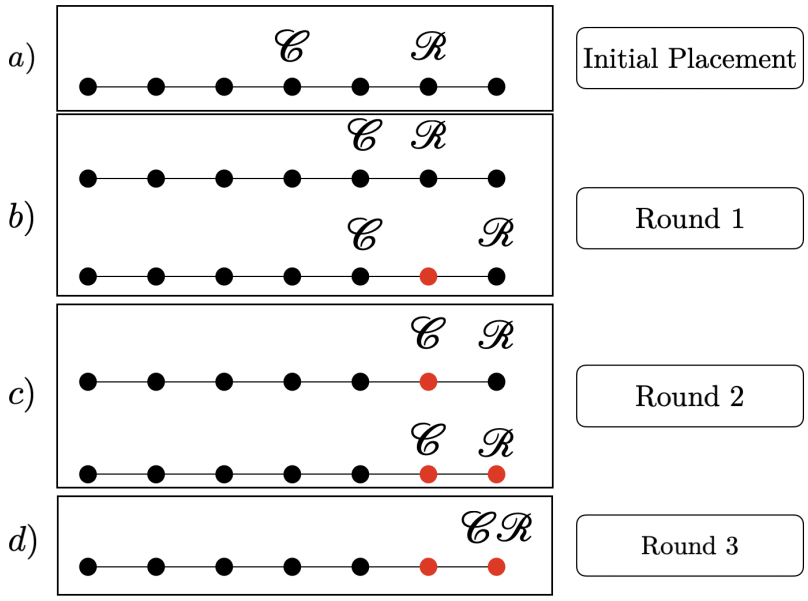


Figure 1.7: The optimal strategy of the cop, $\mathscr{C}$, and the robber, $\mathscr{R}$, on P_7 where red vertices represent damaged vertices.

the robber being able to complete their turn. We will prove in Section 2.2 that this is indeed the optimal strategy for the cop and the robber.

The damage number of a graph can be examined on any graph, using any number of cops. Keeping in mind that the goal of the cops is no longer to capture the robber, the question can be posed even when there are not enough cops to guarantee the capture. In [8], Cox and Sanaei focus on the damage number of a graph with one robber and one cop, even if the graph is not cop-win. We will also only consider the game using one cop and one robber.

1.5.2 Formal Definitions and Results in Literature

In this section, we will formally define the damage number of a graph and summarize initial results provided in [8]. We begin by defining how damage is inflicted on a graph.

Definition 1.6 ([8]). *We say a vertex u is **damaged** by the robber if the robber occupies u in round $i \geq 0$ and moved to a neighbouring vertex in the next round.*

Any vertex that has not been damaged is *undamaged*. Figure 1.8 illustrates how a robber damages an arbitrary vertex v_j. Part a) features the robber on position v_j at round $i \geq 0$. In Part b), the robber then moves to the neighbouring vertex v_{j+1} at

round $i+1$. At this point the vertex v_j is damaged, signified by the vertex v_j changing colour to red in the figure. Part c) illustrates how the robber can still access vertex v_j on round $i+2$, despite its endured damage. In the process of moving back to v_j, the robber also damages v_{j+1}. Notice how damage is non-compounding, meaning vertex v_j cannot be damaged a second time by the robber on round $i+3$, even if they are not captured by the cop on this round. Once a vertex has been damaged it will remain damaged permanently and cannot be further damaged. Furthermore, since we are considering reflexive graphs, if the robber is on vertex v on round i and remains on vertex v for round $i+1$ without being captured, then the vertex v is damaged in between round i and $i+1$. Understanding how a vertex can be damaged on a graph, we can now formally define our parameter of interest, the damage number of a graph.

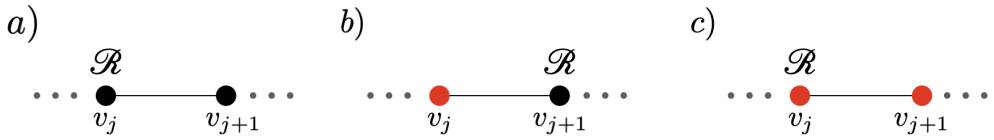


Figure 1.8: How a robber damages a vertex.

Definition 1.7 ([8]). *The **damage number** of a graph G, $dmg(G)$, is the minimum number of unique vertices that can be damaged by the robber.*

We say a cop is *protecting* a vertex (or vertices) if the vertex (vertices) is in the closed neighbourhood of the cop's position. Given a strategy for the cop on a graph, we say that a vertex v is *protected*, if the robber cannot damage v at any point in the game. If a cop *passes in all rounds*, then they remain on their current vertex unless capture is possible, in which they move to an adjacent vertex to capture the robber. As described above, we only consider the game with one cop and one robber.

Since the goal of the cop and the robber differ when considering the damage number of a graph as opposed to the original game of Cops and Robbers, the optimal strategy in both games may differ. In fact, when we consider the damage number of a k-cop win graph with fewer than k cops, since we can guarantee the cops have no strategy to capture the robber on the graph in the original game, we know that the cops will have to develop a new strategy to minimize the damage the robber can inflict on a graph. To explore this idea further, we will consider the cycle graph C_5, shown in Figure 1.9. From our discussion in Section 1.2, we know $c(C_5) = 2$. Therefore, when

considering the damage number of C_5 with an individual cop, capture is not possible, which impacts the strategy of the cop. Without loss of generality, assume the cop places themselves on vertex v_1. To avoid being captured immediately, the robber is restricted to placing themselves on v_3 or v_4. Without loss of generality, assume the robber starts on v_3. To limit the number of vertices the robber can damage, the cop moves to v_2. In response, the robber can move to v_4, damaging vertex v_3 in the process. Following this, the cop will then will move back to v_1 to stop the robber from moving to v_5. The robber will then move back to v_3, thus damaging v_4. Both the cop and robber can repeat these moves indefinitely, so we know $dmg(C_5) = 2$. Through our exploration of C_5, we can see how the cop needs to develop a new strategy, independent to that in the original game, to limit the damage the robber can inflict on the graph. Differences in strategies between the damage number of a graph and the original game can also be seen in cop-win graphs. Even though we can guarantee there is a strategy for the cop to capture the robber, it may not be the best strategy to minimize damage on the graph. We will discuss this further in Section 1.5.3 when discussing cop-win graphs and damage number. Cox and Sanaei generalize the strategy of the cop and robber in C_5 to provide the damage number for all C_n where $n \geq 4$. Given that capture may not be possible on a k-cop win graph, this provides a natural separation in the analysis between cop-win graphs and k-cop win graphs, which we will explore in Section 1.5.3 and Section 1.5.5 respectively.

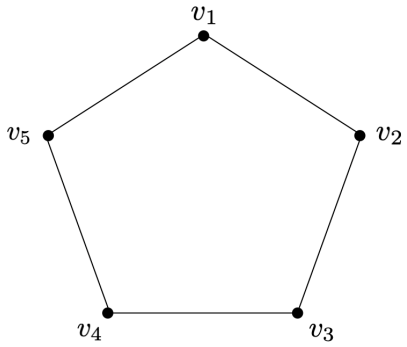


Figure 1.9: The cycle graph C_5.

Theorem 1.8 ([8]). *If $n \geq 4$, $dmg(C_n) = \lfloor \frac{n-1}{2} \rfloor$.*

We now discuss the general upper and lower bound provided by Cox and Sanaei that can be used to bound the damage number of any graph.

Theorem 1.9 ([8]). *If G is a graph with order n, then*

$$\left\lfloor \frac{rad(G)}{2} \right\rfloor \leq dmg(G) \leq n - (\Delta(G) + 1).$$

Suppose the cop starts on the vertex $v \in V(G)$ where $deg(v) = \Delta(G)$. If they pass in all rounds, then they can protect $\Delta(G) + 1$ vertices since $\Delta(G) + 1 = N[v]$, thus giving the upper bound for Theorem 1.9. For the lower bound, assume the cop is on a central vertex and the robber is on a vertex $rad(G)$ away. We know that every path between the cop and robber is at least distance $rad(G)$ in length. Therefore as the cop and robber move towards each other, the robber damages $\left\lfloor \frac{rad(G)}{2} \right\rfloor$ vertices when the players meet in the middle of the path.

Cox and Sanaei also use random graphs to provide an estimate that indicates that many families of graphs have a damage number less than half the order of the graph.

Theorem 1.10 ([8]). *For a graph with order n, with extreme probability, a robber can damage less than $(1 + o(1))\frac{n}{2}$ vertices.*

A $G(n, p)$ model, introduced by Erdös–Rènyi in [10], is a graph created by connecting labeled vertices at random. Each edge is included with a probability of p and the probability of generating a graph with n vertices and M edges is $p^M(1-p)^{\binom{n}{2}-M}$. When $p = \frac{n}{2}$, all $2^{\binom{n}{2}}$ graphs with n vertices are chosen with equal probability. It is known that if $p \in (0, 1)$ then random graph $G \in G(n, p)$ has a vertex of degree $(1 + o(1))pn$, [8]. Since we want graphs to have an equal probability of being generated, $p = \frac{1}{2}$, then G has a vertex of degree $(1 + o(1))\frac{n}{2}$. Thus implying the cop can protect at least $(1 + o(1))\frac{n}{2}$ vertices.

1.5.3 Damage Number of Cop-Win Graphs

In this section, we will discuss the damage number of cop-win graphs. We will start with a simple result on multi-partite graphs and then relate the damage number to the capture time of a graph. Furthermore, we will examine the damage number on a specific graph class, to illustrate an example of when the optimal strategy for the players does not mirror the optimal strategy in the original game, even though capture is possible in the graph.

Lemma 1.11 ([8]). *If G is a complete multi-partite graph, then $dmg(G) \leq 1$.*

If G contains a universal vertex, then $dmg(G) = 0$. Note that if the robber starts on a different partite set than the cop they would be caught immediately; therefore, we can assume the robber starts on the same partite set as the cop. The cop's strategy is to pass in all rounds, unless capture is possible. If the robber moves from their starting vertex, then they must move to a different partition, where capture is possible by the cop in the next round. In such a case, $dmg(G) = 1$.

As previously discussed, damage of a vertex is permanent and cannot be compounded. Given this caveat, we can relate the damage number to other parameters studied in the field. Recall that $capt(G)$ is the number of rounds required for the cop to capture the robber in a cop-win graph G, as defined in Section 1.4.

Lemma 1.12 ([8])**.** *If G is a cop-win graph, then $dmg(G) \leq capt(G) - 1$.*

Since G is a cop-win graph, we know that capture is possible. Additionally, since $capt(G)$ measures the minimum length of the game, the robber can reach at most $capt(G)$ distinct vertices if they move to a new vertex on every round. Let u be the robber's position at round $capt(G)$. If the robber was not on u in round $capt(G) - 1$ then u is not damaged and we are done. If the robber passes on u in round $capt(G)-1$ then u is damaged. Despite this, the robber could have only damaged $capt(G) - 2$ distinct vertices before round $capt(G) - 1$, therefore we can conclude that the robber can only damage $capt(G) - 1$ vertices.

It is useful to bound the damage number of a graph with different parameters relating to Cops and Robbers because we can use results found in these research areas and relate them to the damage number. Recalling Theorem 1.5, on a cop-win graph G with $|V(G)| \geq 5$, since $capt(G) - 1 \leq |V(G)| - 4$ we can also conclude $dmg(G) \leq |V(G)| - 4$. Although Theorem 1.5 did not consider the damage number explicitly, we were able to connect Theorem 1.5 and Lemma 1.12 through the capture time of a graph to provide an upper bound for cop-win graphs of order five or greater.

As described in [8], the bound for Lemma 1.12 is tight for some graphs such as trees and graphs containing a universal vertex. Despite this, Lemma 1.12 is not tight for all graphs. Consider the family of graphs H_n as illustrated in Figure 1.10.

In [11], the authors indicate if $n \geq 7$, then $capt(H_n) = |V(H_n)| - 4$. To carry out their optimal strategy in the original game, the robber visits all but five vertices. However, the damage number of the graph is less than $|V(H_n)| - 4$. Our explanation of this result mimics the proof provided in [8].

Theorem 1.13 ([8])**.** *If $n \geq 7$, then $dmg(H_n) = \left\lfloor \frac{n-3}{2} \right\rfloor - 1$.*

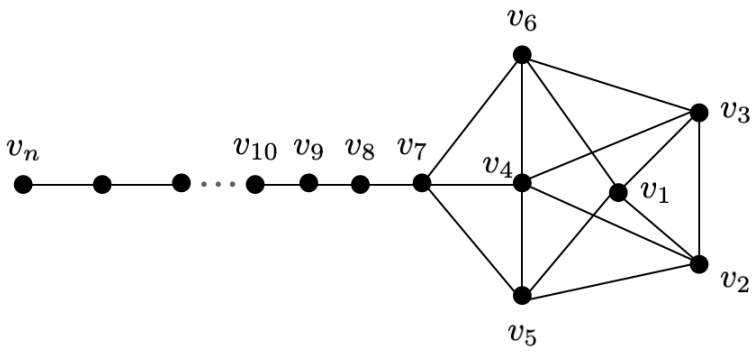


Figure 1.10: A graph H_n from [8] in which $capt(H_n) = |V(H_n)| - 4$.

When $n = 7, 8$ we can place the cop on v_4 and the damage number is one. If $n = 7$, then $N[v_4] = V(G) \setminus \{v_1\}$, making v_1 the only vertex the robber can damage. If $n = 8$, then $N[v_4] = V(G) \setminus \{v_1, v_8\}$, however a player cannot move between v_1 and v_8 without moving to $N[v_4]$.

Suppose now $n \geq 9$. We will consider the possible strategies for the players to conclude that $dmg(H_n) = \lfloor \frac{n-3}{2} \rfloor - 1$.

Assume that both players are of distance two away on the path $v_7 v_8 v_9 \dots v_n$. Note that when the robber exits the path onto v_7, if the cop is right behind the robber then they can only damage four vertices on H_7.

Suppose then the cop places themselves on the vertex $v_{5+\lfloor \frac{n-3}{2} \rfloor - 1}$ on the first round. If the robber places themselves a distance of two away to the right of the cop, then they will damage $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices. If the robber places themselves left of the cop, then they have access to, and will damage, $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices.

What if the cop does not place themselves on the vertex $v_{5+\lfloor \frac{n-3}{2} \rfloor - 1}$? If the cop places themselves on the left of that vertex, then there is a unobstructed path of length $\lfloor \frac{n-3}{2} \rfloor$. Subsequently, if the cop starts on the left of vertex $\lfloor \frac{n-3}{2} \rfloor$, the robber has the ability to damage five vertices in H_7 rather than four. Given these possible situations, we can conclude that $dmg(H_n) = \lfloor \frac{n-3}{2} \rfloor - 1$.

Using Lemma 1.12, we could determine $dmg(H_n) \leq n-4$; however, Theorem 1.13 indicates that $dmg(H_n) = \lfloor \frac{n-3}{2} \rfloor - 1$, which shows Lemma 1.12 is not tight for all cop-win graphs. Furthermore, since Lemma 1.12 is not tight for H_n, it indicates the strategy the cop plays in the original game can differ from the strategy played in this variant even when capture is possible, as alluded to in Section 1.5.2. This proves even in cop-win graphs, because the goal of cop and the robber change in the damage

number variant, we cannot guarantee that the same strategy of the players in Cops and Robbers will be optimal.

1.5.4 Damage Number of Small Graphs

Cox and Sanaei also investigated the damage number for all graphs where $|V(G)| = n \leq 8$ and proved that the damage number of these graphs does not exceed $\lfloor \frac{n}{2} \rfloor$. To accomplish this, they found the maximum damage number for each $n = 1, 2, \ldots, 8$ and provided an example of a graph G in which $dmg(G) = \lfloor \frac{n}{2} \rfloor$. We will now summarize their results.

Theorem 1.14 ([8]). *If G is a graph with order n where $n \leq 8$, then $dmg(G) \leq \lfloor \frac{n}{2} \rfloor$.*

In [8], the authors state that chordal graphs are cop-win graphs. Furthermore, the capture time for a chordal graph is bounded above by $\lfloor \frac{n}{2} \rfloor$ (see [2]). Therefore if G is a chordal graph or $G \cong C_n$, we can apply Lemma 1.12 and Theorem 1.8 to see that $dmg(G) \leq \lfloor \frac{n}{2} \rfloor$. Assume now that G is not chordal and $G \ncong C_n$, then there must exist a vertex $v \in V(G)$ such that $deg(v) \geq 3$. If the cop places themselves on v and passes in all rounds, then they can successfully protect four vertices, namely $N[v]$. However since $n \leq 8$, then $dmg(G) \leq n - 4 \leq \lfloor \frac{n}{2} \rfloor$. Thus we can conclude $dmg(G) \leq \lfloor \frac{n}{2} \rfloor$ when $n \leq 8$.

In [8], Cox and Sanaei investigated the maximum damage number of graphs of order $n = 1, 2, \ldots, 8$. Their findings are illustrated in Table 1.1.

n	Maximum dmg(G)	Example of a graph with maximum damage number
1,2,3	0	Any
4	1	C_4
5	2	C_5
6	2	C_6
7	3	C_7
8	4	The Möbius ladder M_8

Table 1.1: The maximum damage number of small graphs and their associated graphs with $n \leq 8$ ([8]).

For values $1 \leq n \leq 7$, this information is easy to verify directly, as most of cases employ Theorem 1.8. When $n = 8$, the damage number of the Möbius ladder is proved by Theorem 3.2 in [8]. Cox and Sanaei prove this theorem by breaking the strategy of the robber into two steps. First, showing the robber can damage three vertices in

the initial three moves and, second, showing the robber can damage a fourth vertex. Furthermore Table 1.1 shows that Theorem 1.14 is tight for some graphs, such as C_5, C_7, and the Möbius ladder. Looking at the pattern exhibited in Theorem 1.8, Theorem 1.13, and Theorem 1.14, we now ask ourselves if the robber can damage more than half of the vertices in a graph. We will explore this question in the next section.

1.5.5 Damage Number of k-Cop Win Graphs

When considering the goals of each player in the original game, recall that capture is not possible on k-cop win graphs with only one cop. As a result, the upper bound provided in Lemma 1.12 is not applicable for k-cop win graphs. In fact, if $n > 8$, the only bounds on the damage number that have been identified thus far are $\lfloor \frac{rad(G)}{2} \rfloor \leq dmg(G) \leq n - \Delta(G) - 1$ in Theorem 1.9. In this section, we will look at the damage number of k-cop win graphs by providing an example of a graph of order n where $dmg(G) > \lfloor \frac{n}{2} \rfloor$. Additionally, we will summarize results discussed in [8] on strongly regular graphs.

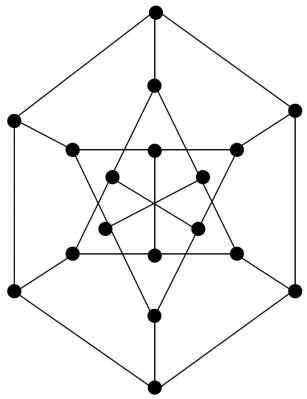


Figure 1.11: The Pappus graph.

The *Pappus* graph, featured in Figure 1.11, is a symmetric, 3-regular, bipartite graph with 18 vertices.

Theorem 1.15 ([8])**.** *The damage number of the Pappus graph is at least 10.*

Cox and Sanaei prove Theorem 1.15 by using the bipartite property of the graph and considering, separately, possible types of cycles the cop could force the robber

through. Since $n = 18$, the damage number of this graph is over half the number of vertices. Theorem 1.15 is an important result, as it proves that not all graphs have a damage number of less than or equal to half their vertices.

Cox and Sanaei also considered the damage number of strongly regular graphs. A *strongly regular graph* is a k-regular graph of order n, where each pair of adjacent vertices have λ common neighbours and every pair of non-adjacent vertices have μ common neighbours. We denote such graphs as $SRG(n, k, \lambda, \mu)$. Perhaps the most common strongly regular graph is $SRG(10, 3, 0, 1)$, also known as the Petersen graph. Figure 1.12 shows the Petersen graph, which we will denote $\mathcal{P}$. Note that $\mathcal{P}$ is a 3-cop win graph. We can check directly that the girth of $\mathcal{P}$ is five, therefore applying Theorem 1.3 we know that $3 \leq c(\mathcal{P})$ and showing $c(\mathcal{P}) \leq 3$ can be done directly.

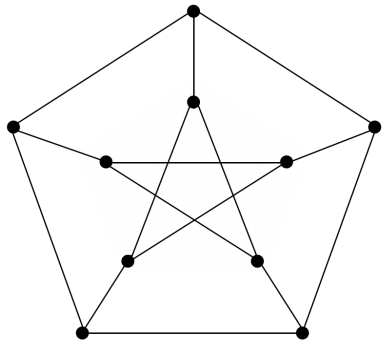


Figure 1.12: The Petersen graph, $\mathcal{P}$.

Since $rad(\mathcal{P}) = 2$ and $\Delta(\mathcal{P}) = 3$, Theorem 1.9 determines that $1 \leq dmg(\mathcal{P}) \leq 6$. If the robber only damages one vertex during the game, then they occupy only one vertex for the entire game or they are captured by the cop in the second round. Let c_i and r_i be the cop and robber's positions respectively for some round i. Since $\mathcal{P}$ is not cop-win, we know there exits a vertex $v_0 \in V(\mathcal{P})$ such that $v_0 \notin N[c_0]$. If $r_0 = v_0$, then the robber will avoid capture on the first round. Notice that for all $c_1 \in N[c_0]$, there always exists some vertex $v_1 \in N[r_0]$ such that $v_1 \notin N[c_1]$. Therefore if $r_1 = v_1$, we know the robber is able to damage r_0 while moving to a vertex that is a distance of at least two away from the cop. As a result, we know that $c_2 \neq r_1$ and the robber can damage another vertex, r_1. As a result, we can guarantee it would not be optimal for the robber to remain on one vertex for the entirety of the game. Looking at the upper bound, if the cop stays on a vertex v where $deg(v) = 3$ for the entirety of the game, the robber can then damage the other six vertices since $V(\mathcal{P}) \setminus N[v]$ is connected.

Having the cop execute a stationary strategy is not effective, which is proven with the following theorem.

Theorem 1.16 ([8]). *The damage number of the Petersen graph is* 5.

Cox and Sanaei proved this result by considering the robber and cop's strategies separately. First, they proved the robber could visit five consecutive undamaged vertices. They then showed that the cop can prevent the robber from damaging a sixth vertex and thus can protect five additional vertices. Cox and Sanaei also contributed to characterizing the damage number of strongly regular graphs by providing a general lower bound. To do so, they considered the last round of the game where the robber cannot cause more damage. Note that K_n satisfies the definition of a strongly regular graph, but contains a universal vertex so $dmg(K_n) = 0$ and thus it is excluded in the bound.

Theorem 1.17 ([8]). *If G is a $SRG(n, k, \lambda, \mu)$ in which $G \ncong K_n$, then $dmg(G) \geq min\{k - \lambda, k - \mu + 1\}$.*

Former literature also uses the damage number of a graph to study different aspects of the game. Cox and Sanaei introduce the damage density of a graph, which is analogous to the cop density. See [4] for information on cop density of a graph and results on the limits of such ratios. In [8], they provide the only known theorem for damage density.

In [6], the authors introduce the damage throttling number of a graph. This parameter aims to optimize the balance between the number of cops being used and the number of vertices being damaged by the robber. See [6], for a formal description of the damage throttling number and the only known bounds associated with this parameter.

Now that we have explored literature focusing on the damage number of a graph, in Chapter 2 we will introduce new results. Our research will focus on formalizing discussion in [8] and providing an improved lower bound for Theorem 1.9. We will also consider new research directions, such as characterizing graphs with damage number one, exploring the relationship between cut-sets and the damage number, and bounding the damage number of various graph products.

Chapter 2

New Results

In Section 2.1, we begin with a discussion of graphs with damage number zero, which is followed by the main result of this section that characterizes graphs with damage number one. Following this, we provide an improved lower bound to that in Theorem 1.9 by describing a new starting position for the robber. Section 2.2 also discusses some graph families for which the improved lower bound is tight. In Section 2.3, we focus our attention on the relationship between the join of graphs and the damage number. We connect cut-sets and the damage number by defining a new property, *star cut-set* in Section 2.4, and show the cop can partition the graph by remaining on a vertex in the star cut-set. We conclude Section 2.4 by providing an upper bound for the damage number of a clique sum graph. Additionally, in Section 2.5 we explore star cut-sets in cactus graphs and provide an upper-bound for the damage number of such graphs.

2.1 Characterization of Graphs with Damage Number One

Cox and Sanaei [8], provide a brief observation about graphs with a damage number of zero. We will now formalize this characterization.

Lemma 2.1. *For a graph G, $dmg(G) = 0$ if and only if G contains a universal vertex.*

Proof. For the backwards direction, assume G is a graph with a universal vertex. If the cop places themselves on a universal vertex, they can capture the robber in one round as every vertex is distance one away from the cop. Therefore if a graph G

contains a universal vertex then $dmg(G) = 0$. For the other implication, we consider the contrapositive and assume that there is no universal vertex in the graph G. Then for all $v \in V(G)$ there exists a vertex $w \in V(G) \setminus \{v\}$ such that $d(v, w) \geq 2$. No matter where the cop places themselves initially, there will always be a vertex at distance two away from the cop. If the robber places themselves on such a vertex initially, they will avoid capture on the first round ensuring $dmg(G) \geq 1$. □

Can we develop a similar characterization for graphs G where $dmg(G) = 1$? We begin with a series of observations about graphs with damage number one that will be used to provide such a characterization.

Remark 2.2. *For a graph G, if $dmg(G) = 1$ then G does not contain a universal vertex.*

From Lemma 2.1, we know $dmg(G) = 0$ if and only if G contains a universal vertex, therefore if $dmg(G) = 1$ then G cannot contain a universal vertex. This means that for all $v \in V(G)$ there exists a vertex $w \in V(G) \setminus \{v\}$ such that $d(v, w) \geq 2$. We will use this property of graphs with damage number one in later proofs to indicate that the robber will always be able to survive the first round of the game.

Lemma 2.3. *Let G be a graph. If $dmg(G) = 1$, there is no benefit for the robber to move during the game.*

Proof. Since $dmg(G) = 1$, if the robber is not captured by the cop for the entirety of the game then they must remain on one vertex. Otherwise, the robber could move between two vertices damaging both in the process. If the robber is caught by the cop, even if the robber decides to move onto another vertex, the robber will be caught in the next move and therefore cannot damage the vertex they moved to. As a result, it is strategically equivalent for the robber not to move, as in either case $dmg(G) = 1$. □

Lemma 2.3 allows us to assume without loss of generality, for graphs with damage number one, the robber remains on their starting vertex for the entirety of the game. Knowing the location of the robber for the entire game will help simplify the explanation of the cop's strategy.

Lemma 2.4. *For a graph G, if $dmg(G) = 1$, then $rad(G) = 2$.*

Proof. For contradiction, assume that $rad(G) \neq 2$. If $rad(G) = 0$, then G is an isolated vertex, in which case $dmg(G) = 0$. If $rad(G) = 1$, then G would contain a universal vertex, which contradicts Remark 2.2.

Assume now $rad(G) = k$ where $k \geq 3$. Let v be any vertex with minimum eccentricity in the graph G. We know that there exists a vertex w that is at least a distance of three away from v. Let $v, x_1, x_2, \ldots, x_n, w$ be the path from v to w. If the cop places themselves on v then the robber can place themselves on x_n. No matter where the cop moves on the first round, the robber can avoid capture and move to vertex w on their first turn, therefore damaging vertex x_n in the process. Again, no matter the cop's strategy, the robber can remain on vertex w for round two without being captured in round three, to damage vertex w.

If the cop were to place themselves on any other vertex in $V(G)$, we know that the eccentricity of this vertex is greater than k. The robber can follow a similar strategy by placing themselves on a vertex at least distance k away. As a result, we can conclude the robber is successful in damaging more than one vertex which contradicts $dmg(G) = 1$. □

Note that the converse of Lemma 2.4 is not true. Consider the graph C_5, which was shown earlier in Figure 1.9. We know that $dmg(C_5) = 2$ from Theorem 1.8. By inspection, we can conclude that $rad(C_5) = 2$, because C_5 does not contain a universal vertex and every pair of vertices is at most distance two away. As a result, C_5 is a graph in which $rad(C_5) = 2$ but $dmg(C_5) \neq 1$.

Using Remark 2.2, Lemma 2.3, and Lemma 2.4, we now provide a characterization for graphs with damage number one.

Theorem 2.5. *For a graph G, $dmg(G) = 1$ if and only if $rad(G) = 2$ and there exists $v \in V(G)$ such that for all $w \in V(G) \setminus N[v]$ either:*

1. *$N(w) \subseteq N(v)$ or,*
2. *there exists $s \in N(v)$ such that $N(w) \subseteq N[s]$.*

Proof. For the forward direction, consider the contrapositive. Assume G is a graph where either $rad(G) \neq 2$ or for all $v \in V(G)$, there exists $w \in V(G) \setminus N[v]$ where $N(w) \not\subseteq N(v)$ and for all $s \in N(v)$, $N(w) \not\subseteq N[s]$. We will show that $dmg(G) > 1$.

If $rad(G) = 0$, then G contains a universal vertex which by Lemma 2.1 means $dmg(G) \neq 1$. If $rad(G) = 1$ or $rad(G) > 2$, then by the contrapostive of Lemma 2.4, we know $dmg(G) \neq 1$.

Now, we will assume that $rad(G) = 2$ and show that no matter the positioning of the cop, the robber is still successful in damaging more than one vertex in G. Since $rad(G) = 2$, no matter where the cop places themselves in G, there always is a vertex at least distance two away. Suppose the cop places themselves on any vertex $v \in V(G)$. The robber can then place themselves on any vertex $w \in V(G) \setminus N[v]$. By the way G is defined, we know $N(w) \not\subseteq N(v)$ and for all $s \in N(v)$, $N(w) \not\subseteq N[s]$. On the first round, the cop has a choice to either stay on vertex v or move to some vertex $s_i \in N(v)$ where $i \in \{1, 2, \ldots, k\}$ and $|N(v)| = k$.

Since $N(w) \not\subseteq N(v)$, we know there exists some vertex $t \in N(w)$ where $t \notin N(v)$. If the cop stays on vertex v on the first round, then the robber can move to vertex t, damaging w in the process. Since $t \notin N(v)$, on the second round, the cop is still at least a distance of two away from the robber. No matter where the cop moves on the second round, the robber can damage vertex t by remaining on it for the second round. In this case, the robber has a strategy to damage more than one vertex.

Suppose instead that the cop moved to the vertex s_i on the first round. Given the properties of G, we know there is some vertex $r \in N(w)$ where $r \notin N[s_i]$. Note that r is not necessarily distinct from the vertex t where $t \in N(w)$ but $t \notin N(v)$. On the first round, the robber can move to vertex r and damage their starting position w. At this point, since $r \notin N[s_i]$, the robber is still at least a distance of two away from the cop. Therefore, no matter where the cop moves on the second round, the robber can remain on vertex r, damaging it in the process. Thus, the robber has a strategy to damage more than one vertex if the cop moves to s_i on the first round. Considering both cases together, we can conclude that if the cop places themselves on any vertex in G, the robber has a strategy to damage at least two vertices. As a result, we can conclude that $dmg(G) \geq 2$.

For the backwards direction, assume that G is a graph with radius two where there exists a vertex $v \in V(G)$ such that for all $w \in V(G) \setminus N[v]$ either $N(w) \subseteq N(v)$ or there exists $s \in N(v)$ such that $N(w) \subseteq N[s]$. We will show that if the cop places themselves on vertex v, they have a strategy to protect $|V(G)| - 1$ vertices and the robber has a strategy to damage one vertex. Since $rad(G) = 2$, we know there is no universal vertex in G and by Lemma 2.1 we know $dmg(G) > 0$.

If the cop places themselves on vertex v, then the cop is protecting $N[v]$ on the first round. Therefore the robber will place themselves on some vertex $w \in V(G) \setminus N[v]$. By the way we have defined G, we know that either $N(w) \subseteq N(v)$ or there exists $s \in N(v)$ such that $N(w) \subseteq N[s]$. If $N(w) \subseteq N(v)$, the cop can protect $V(G) - 1$

vertices by passing in all rounds. Since $N(w) \subseteq N(v)$, the robber cannot move off of vertex w without being captured, which is strategically equivalent by Lemma 2.3. Suppose then $N(w) \not\subseteq N(v)$ but there exists $s \in N(v)$ such that $N(w) \subseteq N[s]$. On the first round, the cop must move to s, otherwise the robber can damage more than one vertex. Then on the robber's turn they can either remain on vertex w or move to some vertex $t \in N(w)$. In both cases the robber damages vertex w. The cop can then protect the remaining $V(G) - 1$ vertices by passing in all rounds.

Therefore the cop has a strategy to protect $V(G) - 1$ vertices. If the robber places themselves on a vertex $w_1 \in V(G) \setminus N[v]$ where $N(w_1) \subseteq N(v)$ then the robber is restricted to remain on w_1 for the entirety of the game. If the robber places themselves on a vertex $w_2 \in V(G) \setminus N[v]$, where there exists an $s \in N(v)$ such that $N(w_2) \subseteq N(s)$ then the robber can remain on vertex w_2 or move to any vertex in $N(w_2)$ and be caught on the next round. Therefore the robber has a strategy to damage one vertex and $dmg(G) = 1$. □

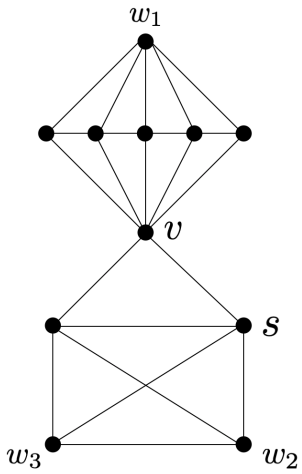


Figure 2.1: A graph G where $dmg(G) = 1$.

The above proof outlined two different strategies for the cop depending on the starting vertex of the robber. Assuming the cop starts on vertex v, as described in Theorem 2.5, if the robber starts on a vertex w such that $N(w) \subseteq N(v)$, the cop wants to remain on vertex v for the entirety of the game. However, if w has the property that there exists a vertex $s \in N(v)$ such that $N(w) \subseteq N[s]$, then the cop must move off vertex v in order to limit the robber's damage. We will now provide an example of a graph with damage number one to illustrate the differences between the

strategies of the cop and robber and examine what these different graph structures may look like.

Consider the graph in Figure 2.1, and suppose the cop places themselves on vertex v. Note that $\{w_1, w_2, w_3\} = V(G) \setminus N[v]$ therefore, we will consider, separately, each of these vertices as starting positions for the robber.

Suppose the robber places themselves on vertex w_1. Notice that vertex w_1 has the property that $N(w_1) \subseteq N(v)$. In this case, the cop's strategy is to remain on vertex v as the robber cannot move from vertex w_1 without being captured in the next round. If the cop moves to some neighbour $u \in N(v)$ to catch the robber, then robber could move to some other vertex $u' \in N(w_1)$ where $u \not\sim u'$. In such a case, since $u \not\sim u'$, the cop cannot capture the robber in the subsequent round and the robber can damage more than one vertex. As a result, we can conclude that it is optimal for the cop to remain on vertex v when the robber chooses w_1 as their starting position.

Now suppose the robber placed themselves on vertex w_2 or w_3 on their first turn. Note that $w_2 \in N(w_3)$ and $w_3 \in N(w_2)$ but $w_2, w_3 \notin N(v)$ therefore $N(w_2) \not\subseteq N(v)$ and $N(w_3) \not\subseteq N(v)$. Consider the closed neighbourhood of vertex s, we can see that $N(w_2) \subseteq N[s]$ and $N(w_3) \subseteq N[s]$. Therefore both w_2 and w_3 satisfy the second condition in Theorem 2.5. Without loss of generality, assume the robber starts on vertex w_2. The cop then wants move from vertex v to vertex s on the first round, since $N(w_2) \subseteq N[s]$. If the robber leaves vertex w_2, then they will be captured on the next round because the cop is occupying a vertex adjacent to every neighbour of w_2. If the robber remains on vertex w_2, then they are only able to damage at most one vertex; furthermore, since $w_2 \in N[s]$ the cop can also move to capture the robber if the robber remains on vertex w_2. As opposed to when the robber started on vertex w_1, the cop needs to move to vertex s to ensure the robber could not damage more than one vertex. Suppose the cop remained on vertex v for the first round. The robber could then move to vertex w_3 damaging w_2 in the process. Since $w_3 \notin N[v]$, the robber is still a distance of two away from the cop. In the following round, the robber can damage vertex w_3 by remaining stationary. As a result, the robber can damage more than one vertex if the cop does not move to s on the first round. No matter where the robber's initial starting position is, they are still able to inflict damage to a vertex, as Theorem 2.5 suggests; however, the cop's strategy during the game varies depending on the starting position of the robber.

2.2 Improved Lower Bound

Theorem 1.9 presented the lower bound given by Cox and Sanaei which indicated that $dmg(G) \geq \lfloor \frac{rad(G)}{2} \rfloor$. In Theorem 2.6, we provide a new starting position for the robber that allows them to damage more vertices than $\lfloor \frac{rad(G)}{2} \rfloor$. Following this, we show that both Theorem 2.6 and Lemma 1.12 are tight for trees. We also provide a connection to the improved lower bound and results found in Section 2.1. Finally, we conclude this section with a discussion of two additional infinite classes of graphs, constructed using graph products, that are also tight with respect to Theorem 2.6. We will define each graph product in this section.

As discussed in Section 1.5.2, the strategy to protect $n - \lfloor \frac{rad(G)}{2} \rfloor$ vertices is carried out by the cop placing themselves on the center of the graph and the robber starting at a vertex $rad(G)$ away. As the cop and robber move towards each other, the cop aims to capture the robber while the robber is aims to damage vertices, they meet at the center of the path. Consider the graph G in Figure 2.2.

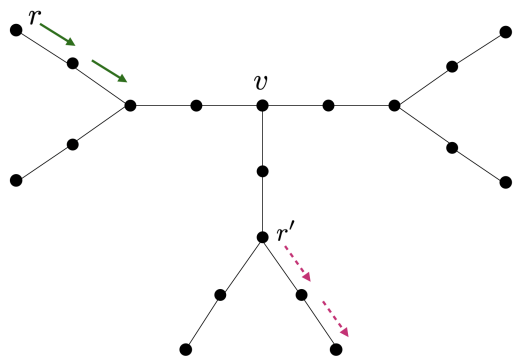


Figure 2.2: A graph G with two different starting positions for the robber, r and r'.

Theorem 1.9 has the cop starting on vertex v, since the eccentricity of v is $rad(G)$, and the robber starting on vertex r, since $d(v, r) = rad(G)$. The solid, green arrows in Figure 2.2 indicate the direction the robber moves along the path. Following this strategy, the cop and robber meet in the middle of the shortest path between v and r and the robber is successful in damaging two vertices in G. However, if we were to place the robber on vertex r', which is distance two away from v, the robber can follow the dashed red arrows and damage three vertices in G. This is a more advantageous starting position for the robber because they can damage more vertices in the graph. We now generalize this idea in the following theorem.

Theorem 2.6. *For any graph G, $dmg(G) \geq rad(G) - 1$.*

Proof. We will consider the possible starting positions of the cop and determine that the robber will have a strategy to damage at least $rad(G) - 1$ vertices no matter the positioning of the cop. Let S be the set of vertices that belong to the center of a graph. Note that every vertex in S has an eccentricity of $rad(G)$. This means that for every $s \in S$ and $v \in V(G)$ that $d(v, s) \leq rad(G)$. Suppose the cop places themselves on some vertex $s \in S$. Let $g \in V(G)$ be a vertex such that $d(g, s) = rad(G)$. Let $s, v_1, v_2, \ldots, v_k, g$ be an isometric path from s to g where $k = rad(G) - 1$. Suppose the robber places themselves on vertex v_2. The robber can then follow the path from v_2 to g, without being captured. Once the robber is on vertex g, the robber will have damaged $rad(G) - 2$ vertices. Note that when the robber first reaches g, the cop will be at least distance two away. At most, if the cop followed the same path as the robber, they will be on vertex v_{k-1} when the robber reaches g. As a result, the robber can remain on g for an additional round and damage it in the process. Thus the robber is successful in damaging $rad(G) - 1$ vertices.

Suppose now the cop places themselves on any vertex $s' \in V(G) \setminus S$. Since $s' \notin S$ then we know that the eccentricity of s' is larger than $rad(G)$. Therefore there exists a vertex $g' \in V(G)$ such that $d(g', s') > rad(G)$. Let $s', v_1, v_2, \ldots, v_k, g'$ be a shortest path from s' to g' where $k > rad(G) - 1$. If the robber places themselves on the vertex v_2 on the first round, they will not be captured by the cop since $v_2 \notin N[s']$. As a result, the robber can move along the path from v_2 to g'. Similarly, when the robber reaches g', the cop is at least distance two away. The robber can then remain on g' for an additional round and the robber can damage at least $rad(G)$ vertices.

Therefore, no matter the positioning of the cop, the robber has a strategy to damage at least $rad(G) - 1$ vertices, thus $dmg(G) \geq rad(G) - 1$, as required. □

Using Theorem 2.6, we can conclude that for all trees T, $dmg(T) \geq rad(T) - 1$. Furthermore, in [8], Cox and Sanaei indicate that Lemma 1.12, is tight for trees. Recall that Lemma 1.12 states that for a cop-win graph G we know that $dmg(G) \leq capt(G) - 1$. Given this, we will provide the intermediate step of showing $capt(T) \leq rad(T)$, which will confirm that $dmg(T) = rad(T) - 1$.

Lemma 2.7. *For any tree T, $capt(T) \leq rad(T)$.*

Proof. To show that $capt(T) \leq rad(T)$, we will show that the cop has a strategy to capture the robber in $rad(T)$ moves. Suppose the cop places themselves on a

vertex s, where s is a vertex in the center of a tree. As a result, we know that the eccentricity of the vertex s is $rad(T)$ so for every vertex $v \in V(T)$, $d(v, s) \leq rad(T)$. Suppose the robber places themselves on some vertex v on the graph, they will be distance at most $rad(T)$ away from the cop. The cop can then take the shortest path from s to v to capture the robber, which will take $rad(T)$ or less rounds. Therefore $capt(T) \leq rad(T)$. □

Now combining results in Lemma 1.12, Theorem 2.6, and Lemma 2.7 gives the following corollary.

Corollary 2.8. *For any tree T, we have that $dmg(T) = rad(T) - 1$.*

Corollary 2.8 confirms that Lemma 1.12 and Theorem 2.6 are tight for some graph classes. Trees are not the only graph class that has a damage number of $rad(G) - 1$. In fact, we can see that Theorem 2.6 is cohesive with findings in Section 1.5.3. Recall Lemma 2.1, which states that for a graph G, $dmg(G) = 0$ if and only if G contains a universal vertex. However, since we are only considering connected graphs, the radius of such a graph must be one, giving us the following corollary.

Corollary 2.9. *For any graph G with a universal vertex, $dmg(G) = rad(G) - 1$.*

From Lemma 2.4, we know that if a graph has a damage number of one, then the graph also has a radius of two. Since we have developed a complete characterization of graphs with damage number one (see Theorem 2.5), this provides an infinite class of graphs for which the damage number is one less than the radius.

Corollary 2.10. *Let G be a graph with damage number one. We then have that $dmg(G) = rad(G) - 1$.*

We can now explore two more infinite graph classes that are tight for Theorem 2.6 but are not guaranteed to have a radius of one or two. These graphs are constructed using two common graph products, which we will define below.

Definition 2.11. *The **Cartesian product** of two graphs G and H, denoted $G\square H$ is a graph where $V(G\square H)$ is the Cartesian product of the vertex set of each graph G and H, or $V(G) \times V(H)$. Two vertices (a, x) and (b, y) are adjacent in $G\square H$ if and only if $a = b$ and $x \in N(y)$ or $x = y$ and $b \in N(a)$.*

Definition 2.12. *The **strong product** of two graphs G and H, denoted $G \boxtimes H$ is a graph where $V(G \boxtimes H)$ is the Cartesian product of the vertex set of each graph G and H, or $V(G) \times V(H)$. Two vertices (a, x) and (b, y) are adjacent in $G \boxtimes H$ if and only if $a = b$ and $x \in N(y)$, $x = y$ and $b \in N(a)$, or $b \in N(a)$ and $x \in N(y)$.*

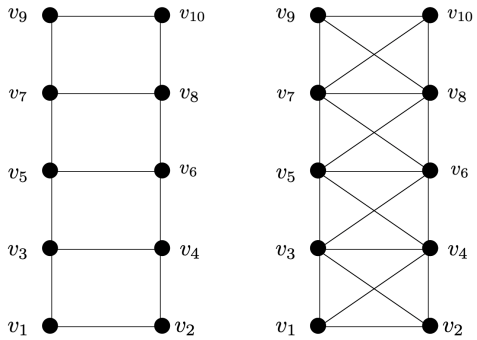


Figure 2.3: $P_2 \square P_5$ and $P_2 \boxtimes P_5$, respectively.

Consider the graph $P_2 \square P_5$ in Figure 2.3, we can see that $rad(P_2 \square P_5) = 3$, since the eccentricity of v_5 is three and v_5 is the center of the graph. Suppose the cop places themselves on vertex v_5 and, without loss of generality, the robber places themselves on some vertex v_i where $i \leq 6$. If $v_i \in N[v_5]$ then the robber will be caught immediately by the cop. If $v_i \notin N[v_5]$, the cop may then move down the path v_5, v_3, v_1 to minimize the damage done by the robber. Suppose we are in the situation where the cop is on vertex v_3 in round ℓ and the robber is on vertex v_2. If the cop was to move to vertex v_1 on round $\ell+1$ the robber could move to v_4 and remain a distance two away from the cop. By doing so, the robber can then move to vertex v_6 on round $\ell + 2$, damaging v_4 in the process. Instead, if the cop remains on vertex v_3 for round $\ell+1$, then the robber cannot move to vertex v_4 on round $\ell+1$ without being captured by the cop in round $\ell + 2$. Additionally, the robber cannot move to any undamaged vertex without entering the cop's neighbourhood. This movement for the cop can be generalized for the strategy of the cop in $P_2 \square P_5$ in any situation where the cop is on vertex v_j and the robber is on vertex v_{j-1}. We can see that the cop cannot simply move down the path $v_j, v_{j-1}, v_{j-2}, \ldots$ to the robber's position in every subsequent round, but instead, needs to consider the robber's position before deciding to move down the path or remain on their current vertex. Given this strategy, the robber is only able to damage two vertices in $P_2 \square P_5$. By Theorem 2.6, we know that this is the minimum number of vertices the robber can damage in the graph.

Now looking at $P_2 \boxtimes P_5$, we can see that $rad(P_2 \boxtimes P_5) = 2$ since the eccentricity of v_5 is two and v_5 is the center of the graph. We can employ a similar strategy for the cop by placing them on vertex v_5 initially. Again, without loss of generality, assume the robber will be captured on some v_i where $i \leq 6$. If $v_i \in N[v_5]$, then the robber is captured. If $v_i \notin N[v_5]$, the cop can move down the path v_5, v_3, v_1 and guarantee the capture of the robber. Note that the cop does not need to pass at any point during their strategy since for any v_j where $j \in \{1, 3, 5\}$, $v_{j-1} \in N[v_j]$, which is not the case for $P_2 \square P_5$. The robber is then only able to damage one vertex in $P_2 \boxtimes P_m$, which is the minimum number of vertices according to Theorem 2.6. We can generalize this strategy for the cop on both graph products, which brings us to the following lemmas, for which the proofs mimic similar strategies.

Lemma 2.13. *Let P_2 and P_m be paths of order two and m respectively, for some $m \in \mathbb{N}$. Then we have that $dmg(P_2 \square P_m) = rad(P_2 \square P_m) - 1$.*

Proof. To show that $dmg(P_2 \square P_m) \leq rad(P_2 \square P_m) - 1$, we will need to show that the cop has a strategy to protect $n - rad(P_2 \square P_m) + 1$ vertices. Let $V(P_2) = \{a, b\}$ and $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. Suppose that the cop initially places themselves on some vertex (y, x_i) such that the eccentricity of (y, x_i) is equal to $rad(P_2 \square P_m)$. Without the loss of generality, assume that $y = a$. The robber can place themselves on any vertex $(t, x_j) \in V(P_2 \square P_m) \setminus (y, x_i)$, where $t \in \{a, b\}$ and $j \neq i$. If $j > i$ then the cop can follow the path $(a, x_{i+1}), (a, x_{i+2}), \ldots, (a, x_{j-1})$. Suppose we are in a situation where the cop is on the vertex (a, x_{i+k}) and the robber is on vertex (b, x_{i+k+1}) for $k \geq 0$. Instead of immediately moving to vertex (a, x_{i+k+1}), the cop should pass. In doing so, the cop is ensuring the robber cannot move to vertex (b, x_{i+k}) without being captured on the next turn. Using this strategy, we can guarantee that every time the cop moves from (a, x_{i+k}) to (a, x_{i+k+1}), they prevent the robber's access to all vertices in the set $\{(\ell, x_p) |\ p \in \{1, 2, \ldots, i + k + 1\}, \ell \in \{a, b\}\}$. Therefore we can assume that the robber's position is now (t, x_j), if $t = a$ then the robber has been caught; if $t = b$ then by definition, $(t, x_j) \in N[(a, x_j)]$ so the robber will be caught in the next move. Similarly, if $j < i$ the cop can follow the path $(a, x_{i-1}), (a, x_{i-2}), \ldots, (a, x_{j+1})$. Again, if we are in a situation where the cop is on vertex (a, x_{i-k}) and the robber is on vertex $(b, x_{i-(k+1)})$. Instead of immediately moving to vertex $(a, x_{i-(k+1)})$, the cop should pass. By doing this, the cop is ensuring the robber cannot move to vertex (b, x_{i+k}) without being captured on the next turn. Therefore every time the cop moves from (a, x_{i-k}) to $(a, x_{i-(k+1)})$, they are preventing the robber's access to vertices in

the set $\{(\ell, x_p)|\ p \in \{i-(k+1), \ldots, m\}, \ell \in \{a, b\}\}$. Since the cop's starting position is on a vertex of minimum eccentricity, we know that (t, x_j) is at most $rad(P_2 \square P_m)$ away. Therefore the cop can protect $n-(rad(P_2 \square P_m)-1)$ vertices. The lower bound follows from Theorem 2.6. □

In the above strategy, for any $x_t \in P_m$, we know that (a, x_t) and (b, x_t) form a cut-set in $P_2 \square P_m$. Since the induced subgraph $P_2 \square P_m[\{(a, x_t), (b, x_t)\}] \cong P_2$, the cop can place themselves on either vertex (a, x_t) or (b, x_t) and restrict the robber from accessing both of these vertices. We can extend this strategy to $P_3 \square P_m$ since for any $x_t \in V(P_m)$ and $P_3 = \{a, b, c\}$ we know vertices (a, x_t), (b, x_t) and (c, x_t) form a cut-set in $P_3 \square P_m$ which can be protected by one cop when they are on vertex (b, x_t). This observation gives us the following lemma, which follows from Lemma 2.13.

Lemma 2.14. *Let P_3 and P_m be paths of order three and m respectively, for some $m \in \mathbb{N}$. Then we have that $dmg(P_3 \square P_m) = rad(P_3 \square P_m) - 1$.*

Proof. We want to show that $dmg(P_3 \square P_m) \leq rad(P_3 \square P_m) - 1$, by proving that the cop has a strategy to protect $n - rad(P_3 \square P_m) + 1$ vertices. Let $V(P_3) = \{a, b, c\}$ where $N[b] = \{a, b, c\}$ but $N[a] = \{a, b\}$ and $N[c] = \{b, c\}$. Furthermore let $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. Suppose that the cop initially places themselves on some vertex (b, x_i) such that the eccentricity of (b, x_i) is equal to $rad(P_3 \square P_m)$. The robber can then place themselves on any vertex $(t, x_j) \in V(P_3 \square P_m) \setminus (b, x_i)$, where $t \in \{a, b, c\}$ and $j \neq i$. If $j > i$ then the cop can follow the path $(b, x_{i+1}), (b, x_{i+2}), \ldots, (b, x_{j-1})$. Suppose we are in a situation where the cop is on vertex (b, x_{i+k}) and the robber is on vertex (a, x_{i+k+1}) or (c, x_{i+k+1}) for some $k \geq 0$. Instead of immediately moving to vertex (b, x_{i+k+1}), the cop should pass on their turn. In doing so, the cop is ensuring the robber cannot move to vertex (a, x_{i+k}) or (c, x_{i+k}) without being captured on the next turn. Using this strategy, we can guarantee that every time the cop moves from (b, x_{i+k}) to (b, x_{i+k+1}), they prevent the robber's access to vertices in the set $\{(\ell, x_p)|\ p \in \{1, 2, \ldots, i+k+1\}, \ell \in \{a, b, c\}\}$. Therefore we can assume that the robber's position is now (t, x_j), if $t = b$ then the robber has been caught, if $t = a$ or $t = c$ then by definition, $(t, x_j) \in N[(b, x_j)]$ so the robber will be caught in the next move. Similarly, if $j < i$ the cop can follow the path $(b, x_{i-1}), (b, x_{i-2}), \ldots, (b, x_{j+1})$. Notice that we are, again, in a situation where the cop is on vertex (b, x_{i-k}) and the robber is on vertex $(a, x_{i-(k+1)})$ or $(c, x_{i-(k+1)})$. Instead of immediately moving to vertex $(b, x_{i-(k+1)})$, the cop should pass. By doing this, the cop is ensuring the robber cannot move to vertex (a, x_{i+k}) or (c, x_{i+k}) without being captured on the

next turn. Therefore every time the cop moves from (b, x_{i-k}) to $(b, x_{i-(k+1)})$, they are preventing the robber's access to vertices in the set $\{(\ell, x_p)|\ p \in \{i-(k+1), \ldots, m\}, \ell \in \{a, b, c\}\}$. Since the cop's starting position is on a vertex of minimum eccentricity, we know that (t, x_j) is at most $rad(P_3 \square P_m)$ away. Therefore the cop can protect $n - (rad(P_3 \square P_m) - 1)$ vertices. The lower bound also follows from Theorem 2.6. $\square$

Recall in our explanation for $dmg(P_2 \boxtimes P_5)$, that we still placed our cop the center vertex of the graph, however the cop does not need to pass during any round of the game. Therefore we can slightly alter the strategy from Lemma 2.13 to have the cop move to a new vertex every round. By doing so, we can provide the damage number for $P_2 \boxtimes P_m$.

Lemma 2.15. *Let P_2 and P_m be paths of order two and m respectively, for some $m \in \mathbb{N}$. Then we have that $dmg(P_2 \boxtimes P_m) = rad(P_2 \boxtimes P_m) - 1$.*

Proof. To show that $dmg(P_2 \boxtimes P_m) \leq rad(P_2 \boxtimes P_m) - 1$, we will need to show that the cop has a strategy to protect $n - rad(P_2 \boxtimes P_m) + 1$ vertices. Let $V(P_2) = \{a, b\}$ and $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. Suppose that the cop initially places themselves on some vertex (y, x_i) such that the eccentricity of (y, x_i) is equal to $rad(P_2 \square P_m)$. Without the loss of generality, assume that $y = a$. The robber can place themselves on any vertex $(t, x_j) \in V(P_2 \boxtimes P_m) \setminus (y, x_i,)$, where $t \in \{a, b\}$ and $j \neq i$. If $j > i$ then the cop can follow the path $(a, x_{i+1}), (a, x_{i+2}), \ldots, (a, x_j)$. Using this strategy, we can guarantee that every time the cop moves from (a, x_{i+k}) to (a, x_{i+k+1}), they prevent the robber's access to vertices in the set $\{(\ell, x_p)|\ p \in \{1, 2, \ldots, i+k+1\}, \ell \in \{a, b\}\}$. Therefore we can assume that the robber's position is now (t, x_j), if $t = a$ then the robber has been caught, if $t = b$ then by definition, $(t, x_j) \in N[(a, x_j)]$ so the robber will be caught in the next move. Similarly, if $j < i$ the cop can follow the path $(a, x_{i-1}), (a, x_{i-2}), \ldots, (a, x_j)$. Therefore every time the cop moves from (a, x_{i-k}) to $(a, x_{i-(k+1)})$, they are preventing the robber's access to vertices in the set $\{(\ell, x_p)|\ p \in \{i-(k+1), \ldots, m\}, \ell \in \{a, b\}\}$. Since the cop's starting position is on a vertex of minimum eccentricity, we know that (t, x_j) is at most $rad(P_2 \boxtimes P_m)$ away. Therefore the cop can protect $n - (rad(P_2 \boxtimes P_m) - 1)$ vertices. The lower bound follows from Theorem 2.6. $\square$

Again, we can extend the strategy for the cop to $P_3 \boxtimes P_k$, since for any $x_t \in V(P_m)$ and $P_3 = \{a, b, c\}$ the vertices (a, x_t), (b, x_t) and (c, x_t) form a cut-set in $P_3 \boxtimes P_m$, which can be protected by one cop on vertex (b, x_t).

Lemma 2.16. *Let P_3 and P_m be paths of order three and m respectively, for some $m \in \mathbb{N}$. Then we have that $dmg(P_3 \boxtimes P_m) = rad(P_3 \boxtimes P_m) - 1$.*

Proof. We want to show that $dmg(P_3 \boxtimes P_m) \leq rad(P_3 \boxtimes P_m) - 1$, by proving that the cop has a strategy to protect $n - rad(P_3 \boxtimes P_m) + 1$ vertices. Let $V(P_3) = \{a, b, c\}$ where $N[b] = \{a, b, c\}$ but $N[a] = \{a, b\}$ and $N[c] = \{b, c\}$. Furthermore let $V(P_m) = \{x_1, x_2, \ldots, x_m\}$. Suppose that the cop initially places themselves on some vertex (b, x_i), such that the eccentricity of (b, x_i), is equal to $rad(P_3 \square P_m)$. The robber can then place themselves on any vertex $(t, x_j) \in V(P_3 \square P_m) \setminus (b, x_i)$, where $t \in \{a, b, c\}$ and $j \neq i$. If $j > i$ then the cop can follow the path $(b, x_{i+1}), (b, x_{i+2}), \ldots, (b, x_j)$. Using this strategy, we can guarantee that every time the cop moves from (b, x_{i+k}) to (b, x_{i+k+1}), the prevent the robber's access to all vertices in the set $\{(\ell, x_p) |\ p \in \{1, 2, \ldots, i+k+1\}, \ell \in \{a, b, c\}\}$. Therefore we can assume that the robber's position is now (t, x_j), if $t = b$ then the robber has been caught, if $t = a$ or $t = c$ then by definition, $(t, x_j) \in N[(b, x_j)]$ so the robber will be caught in the next move. Similarly, if $j < i$ the cop can follow the path $(b, x_{i-1}), (b, x_{i-2}), \ldots, (b, x_j)$. Therefore every time the cop moves from (b, x_{i-k}) to $(b, x_{i-(k+1)})$, they are preventing the robber's access to vertices in the set $\{(\ell, x_p) |\ p \in \{i - (k+1), \ldots, m\}, \ell \in \{a, b, c\}\}$. Since the cop's starting position is on a vertex of minimum eccentricity, we know that (t, x_j) is at most $rad(P_3 \boxtimes P_m)$ away. Therefore the cop can protect $n - (rad(P_3 \boxtimes P_m) - 1)$ vertices. The lower bound also follows from Theorem 2.6. □

Lemma 2.13, Lemma 2.14, Lemma 2.15 and Lemma 2.16 provide infinite graph classes for which Theorem 2.6 is tight, however this is not a complete characterization of such graphs.

2.3 Graph Joins

In this section, we will determine the damage number for graph joins, which is dependant on the damage number of the input graphs. We begin by defining how the join of a graph connects each input graph.

Definition 2.17. *The **join** $G \vee J$ of graphs G and J connects the vertices of G with the vertices of J. This means for all $v \in V(G)$, $v \sim w$ for all $w \in V(J)$.*

Note that the join $G \vee J$ is connected, regardless of whether G or J are connected (see Figure 2.4). As a result, in this section, we will not assume that either graphs G or J in $G \vee J$ are connected.

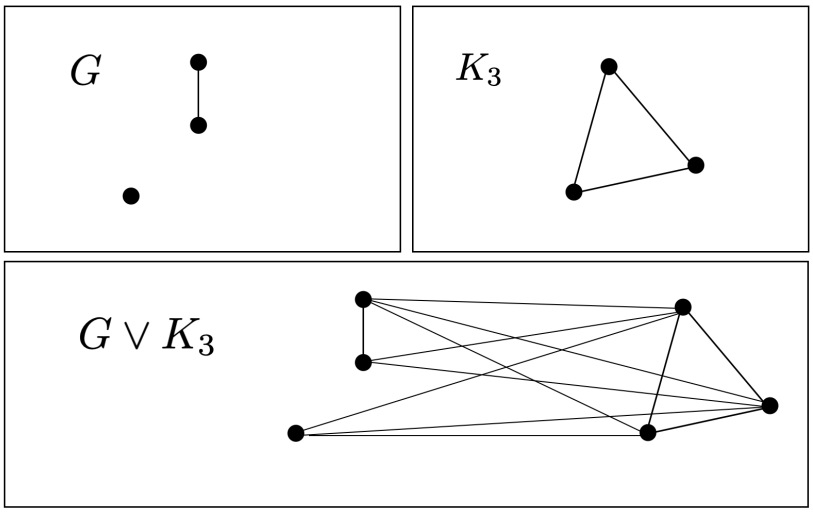


Figure 2.4: A disconnected graph G, the complete graph K_3, and the join $G \vee K_3$.

Lemma 2.18. *If either graphs G or J contain a universal vertex, then $dmg(G \vee J) = 0$.*

Proof. Without loss of generality, assume that G has a universal vertex, u. As a result we know that $u \sim v$ for all $v \in V(G)$. By the definition of a graph join, we know that $u \sim w$ for all $w \in V(J)$. Therefore, u is universal in $G \vee J$ for any graph J, and the result follows. □

Theorem 2.19. *For any graphs G and J, $dmg(G \vee J) = \min\{dmg(G), dmg(J)\}$.*

Proof. Suppose $dmg(G) = k$ and $dmg(J) = \ell$ and assume without loss of generality that $k \leq \ell$. First we will show that $dmg(G \vee J) \leq \min\{dmg(G), dmg(J)\}$ by showing the cop has a strategy to protect $|V(G \vee J)| - k$ vertices. The cop can begin by placing themselves on a vertex in G. If G contains a universal vertex then by Lemma 2.18, $dmg(G \vee J) = 0$ and the cop has a strategy to protect all vertices. Assume then that G does not contain a universal vertex, and the robber places themselves on a vertex in G to avoid immediate capture. If the cop remains on G for the remainder of the game, they are able to protect all vertices in J because for all $v \in V(G)$, $V(J) \subseteq N[v]$. Therefore, if the robber moved to a vertex in J, they would be caught in the next round, resulting in the robber damaging less vertices than if they stayed in G. As a result, the cop player has a strategy for protecting $|V(G \vee J)| - k$ vertices.

Now we will prove that $dmg(G \vee J) \geq \min\{dmg(G), dmg(J)\}$ by showing that the robber has a strategy to damage k vertices. To do so, we will show that if the

cop moves from G to J at any point in the game the robber can damage more than k vertices. Suppose the cop places themselves on G at the start of the game. First notice that if either G or J contain universal vertices then $dmg(G \vee J) = 0$ by Lemma 2.18. Therefore, we assume that G and J do not contain universal vertices.

Let i be the first round the cop moves from G to J. Suppose they move to some vertex $v \in V(J)$. Therefore the robber managed to damage at most $i-1$ vertices where $1 \leq i < k$. (We know that $i \neq k$ or else the cop would not leave G at this point). In round i, the robber will also need to move to J to avoid capture in the next round. Assume then the robber moves to some vertex $w \in V(J)$ where $w \notin N[v]$. We know such a w exists since J does not contain a universal vertex.

Now considering round $i+1$, the robber has already damaged at most i vertices. We know that the cop cannot capture the robber on the next round as $w \notin N[v]$. Therefore the cop has two options, to either stay in J or move back to G.

Suppose the cop moves back to G, then to avoid capture in the next round, the robber must also move to G which damages w in the process. No matter the positioning of the cop, the robber could continue to damage $k-i$ vertices in G, since $dmg(G) = k$. Therefore the robber would damage at least $k+1$ vertices, including the vertex w that is damaged in J. Suppose that the cop remains in J for the rest of the game then the robber could damage ℓ vertices in J. However then the robber could have damaged at most $\ell + i$ vertices. Since $i \geq 1$ and $\ell \geq k$ then we can conclude $\ell + i > k$. Therefore the robber can damage more than k vertices.

Since the robber has a strategy for damaging more than k vertices if the cop leaves G, then we know the cop will remain in G for the entirety of the game. Therefore $dmg(G) \geq k$, as required. □

2.4 Star Cut-Set

We now consider the relationship between cut-sets and the damage number of a graph. We define a star cut-set, star cut area, and the star cut component number to determine an upper bound on the damage number for graphs with star cut-sets.

Definition 2.20. *A set of vertices, S is a **star cut-set** of a graph G if $G[S]$ has a spanning subgraph isomorphic to $K_{1,n}$, for some $n \in \mathbb{Z}^+ \cup \{0\}$, and S is a cut-set.*

We will discuss a strategy for a cop to occupy a single vertex of a star cut-set to force the robber onto a partitioned set of vertices in the graph. If a cut-set can

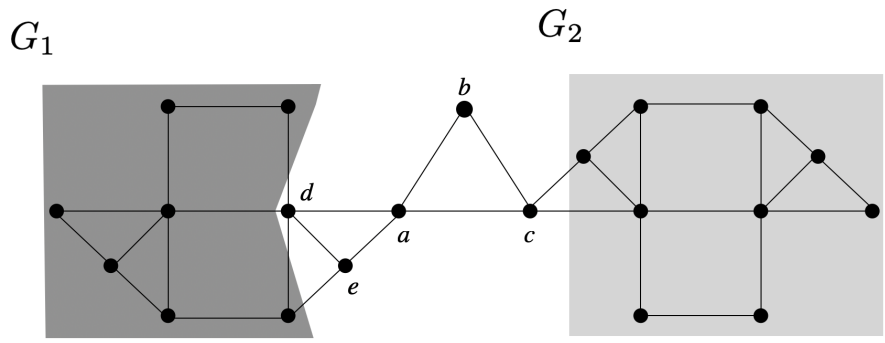


Figure 2.5: A graph G with star cut-set $\{a, b, c, d, e\}$ with connected components G_1 and G_2.

be guarded by one cop, the cop can protect the vertices of the components that the robber does not place themselves in. For example, consider the graph in Figure 2.5, note that $N[a] = \{a, b, c, d, e\}$ which is also a cut-set and thus a star cut-set. Therefore if we place a cop on vertex a they can protect vertices $\{a, b, c, d, e\}$. The robber is then forced to place themselves on some vertex in $V(G) \setminus \{a, b, c, d, e\}$. As long as the cop remains on vertex a, the robber cannot access both components G_1 and G_2, as the robber would need to move to $N[a]$ which the cop is protecting on vertex a. By remaining on a single vertex, the cop is able to protect more than $(\delta(G) + 1)$ vertices from damage in G, which is the damage number of the cop's stationary strategy in the proof of Theorem 1.9. We will now generalize this strategy and present the results for in this section in multiple steps.

Lemma 2.21. *Suppose G is a graph with star cut-set S. Let $G_1, \ldots, G_k$ be the connected components of $G[V(G) \setminus S]$ for some $k \geq 2$. The cop can protect at least $|V(G)| - \max_{1 \leq i \leq k} |V(G_i)|$ vertices in G.*

Proof. Let G be a graph with a star cut-set S. Suppose that $G[V(G) \setminus S]$ consists of connected components $G_1, \ldots, G_k$ for some $k \geq 2$. Furthermore set $\ell = \max_{1 \leq i \leq k} |V(G_i)|$ and let $|V(G_t)| = \ell$ for some $t \in \{1, 2, \ldots, k\}$. Suppose the cop places themselves on a universal vertex in S, that is, any vertex $c \in S$ such that $d(c, v) \leq 1$ for all $v \in S$. The robber is then restricted to place themselves on some vertex $r \in V(G) \setminus S$ since $N[c] = S$. However since S is a star cut-set, we know that the graph $G[V(G) \setminus S]$ results in connected components of G. Suppose the robber is in component G_j for some $j \in \{1, 2, \ldots, k\}$. Assuming the cop is still on vertex c, if the robber attempted to leave G_j they would have to enter the neighbourhood of the cop. As a result, the

robber is restricted to moving to vertices in G_j. If the cop then remains on vertex c for the rest of the game and the robber can proceed to damage all of the vertices in G_j. Considering the robber wants to damage as many vertices as possible, we can determine that $G_j = G_t$ since $|V(G_t)| = \ell = \max_{1 \leq i \leq k} |V(G_i)|$. □

Lemma 2.21 provides a clear strategy for the cop if a graph G contains a unique star cut-set. As shown in Figure 2.6, the cop can only start on vertex c to execute the strategy in Lemma 2.21, since $\{c\}$ is the only star cut-set in this graph. Despite this, Lemma 2.21 provides no direction for the starting position of the cop if a graph contains multiple star cut-sets, as seen in Figure 2.5 where $N[a]$ and $N[c]$ form star cut-sets in G. We introduce the following definitions to better establish the starting position of the cop to maximize the number of vertices they can protect.

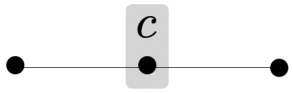


Figure 2.6: A graph H with a unique star cut-set.

Definition 2.22. *The **star cut area** of a star cut-set S of a graph G is the cardinality of the largest component in $G[V(G) \setminus S]$ and is denoted $\mu(S)$. The **star cut component number** of a graph G, denoted $M(G)$, is the minimum star cut area over all star cut-sets in G, or*

$$M(G) = \min\{\mu(S) : S \subseteq V(G),\ S \text{ is a star cut-set}\}.$$

Theorem 2.23. *Let G be a graph with a star cut-set, S, where $\mu(S) = M(G)$. We have that $dmg(G) \leq M(G)$.*

Proof. Let c be a vertex such that $c \in S$ and $d(c, v) \leq 1$ for all $v \in S$. Suppose the cop starts by placing themselves on vertex c. Let $G_1, G_2, \ldots, G_k$ be the connected components of $G[V(G) \setminus S]$. We know that by Lemma 2.21, that the cop can protect at least $|V(G)| - \max_{1 \leq i \leq k} |V(G_i)|$ vertices in G because the robber cannot access other components in $G[V(G) \setminus S]$ if the cop remains on c, as the proof indicates. However,

by the way S is defined, we know that $\mu(S) = M(G)$ and can therefore guarantee that $|V(G)| - \max_{1 \leq i \leq k} |V(G_i)|$ is maximized. From here, we can conclude that the robber has the ability to damage at most $M(G)$ vertices. □

To illustrate the usefulness of a star cut-set, we will consider another graph product.

Definition 2.24. *Let G and H be graphs with cliques $\{v_1, \ldots, v_k\}$ and $\{u_1, , \ldots, u_k\}$, respectively. The **clique sum** of G and H, denoted $G \oplus H$, is the graph formed by identifying u_i with v_i for each $1 \leq i \leq k$.*

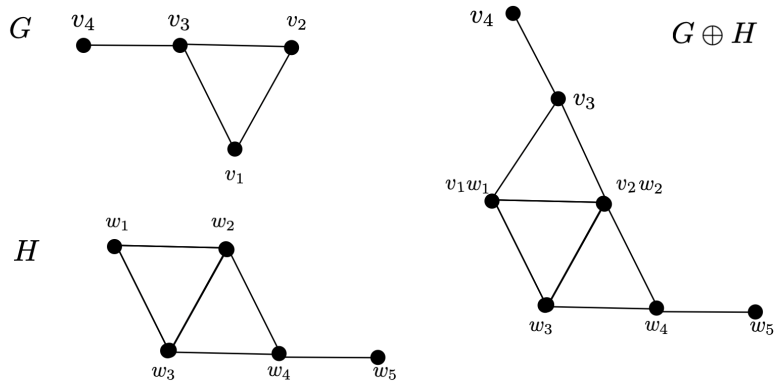


Figure 2.7: A graph G and H, with clique sum $G \oplus H$. A vertex $v_i w_i \in V(G \oplus H)$ represents the identification of vertex v_i and w_i for some $i \in \{1, 2\}$.

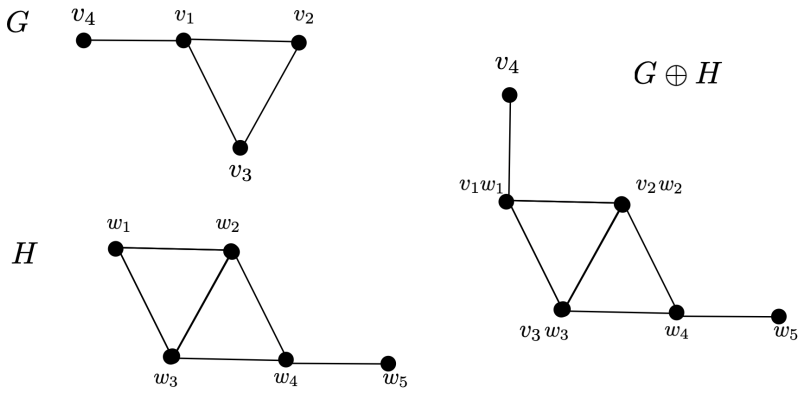


Figure 2.8: A graph G and H, with clique sum $G \oplus H$. A vertex $v_i w_i \in V(G \oplus H)$ represents the identification of vertex v_i and w_i for some $i \in \{1, 2, 3\}$.

Note that the clique sum of a graph is not necessarily unique and depends on the clique chosen and the labeling of each vertex. For example consider Figure 2.7 and Figure 2.8. Notice that in each figure, G and H are the same graph; however, we can clearly see that their clique sums are not isomorphic, since each graph contains a different number of vertices. As we identified different orders of cliques for each clique sum, our resulting graphs must have different order since $|V(G \oplus H)| = |V(G)| + |V(H)| - k$ where k is the order of the clique being identified. Variations in our clique sum can also result from different labeling of vertices in the identified clique of the clique sum.

Note that since a clique sum identifies vertices of cliques, it ensures there is a clique of order k in the graph $G \oplus H$. Suppose that S is a clique of order k in $G \oplus H$. If $G \oplus H[V(G \oplus H) \setminus S]$ forms at least two connected components in G, then we know the clique, S, forms a star-cut set in the graph. Considering the graph $G \oplus H$ in Figure 2.7, we can see that the vertices identified by the clique sum, v_1w_1 and v_2w_2, form a star cut-set with connected components $G \oplus H[\{v_3, v_4\}]$ and $G \oplus H[\{w_3, w_4, w_5\}]$. Similarly, $G \oplus H$ in Figure 2.8, v_1w_1, v_2w_2, and v_3w_3 form a star cut-set where the two connected components are $G \oplus H[\{v_4\}]$ and $G \oplus H[\{w_4, w_5\}]$. Therefore, we can place the cop on one vertex in the identified clique and protect vertices in one of the connected components, which gives us the following corollary.

Corollary 2.25. *Let G and H be graphs where $|V(G)|, |V(H)| > k$. If G and H contain cliques $\{v_1, v_2, \ldots, v_k\}$ and $\{u_1, u_2, \ldots, u_k\}$ in G and H respectively then*

$$dmg(G \oplus H) \leq \max\{|V(G)|, |V(H)|\} - k.$$

Figure 2.9 and Figure 2.10 illustrate the cop's positioning in $G \oplus H$ to protect $\max\{|V(G)|, |V(H)|\} - k$ vertices. A red vertex in Figure 2.10 indicates endured damage and $\mathscr{C}$ is the positioning of the cop for the entirety of the game, unless capture is possible inside of the identified clique. Suppose the cop is on a vertex $v_iw_i \in V(G \oplus H)$, where $v_i \in S_G$ and $w_i \in S_H$ for identified cliques S_G and S_H in $G \oplus H$. If there exists a vertex $u \in N(v_iw_i)$ such that $u \in S_G$ and $u \notin S_H$, or conversely, then we know the robber can damage this vertex if the cop is executing the strategy outlined by Corollary 2.25. Although $u \in N[v_iw_i]$, since it is not a vertex in the identified cliques then, the cop does not move to vertex u even though capture is possible. Vertex w_3 in Figure 2.9 has this property, where if the robber is on vertex w_3, even though they are a distance of one away from the cop, the cop will remain on

v_1w_1 instead of moving to capture. As opposed to vertex w_3, since the vertex v_2w_2 is part of the identified clique, if the robber moves to this vertex at any point in the game the cop will move to capture the robber since $v_2w_2 \in N[v_1w_1]$.

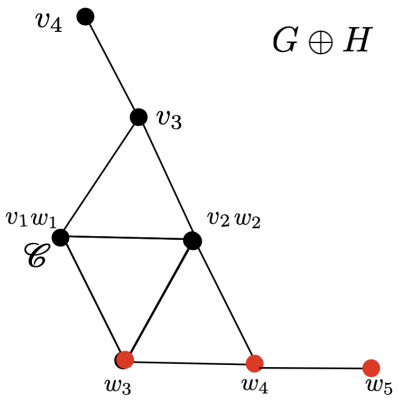


Figure 2.9: The vertices that may be damaged by the robber in the graph $G \oplus H$ from Figure 2.7 where the cop remains stationary on vertex v_1w_1, unless capture is possible in the identified clique.

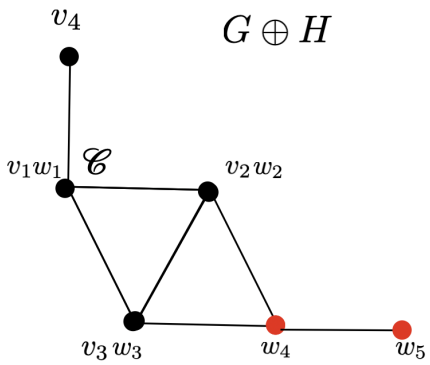


Figure 2.10: The vertices damaged by the robber in the graph $G \oplus H$ from Figure 2.8 where the cop remains stationary on vertex v_1w_1 unless capture is possible in the identified clique.

Let $\{v_1, v_2, \ldots, v_p\}$ and $\{w_1, w_2, \ldots, w_p\}$ be maximal spanning cliques of the same order in G and H respectively. Suppose that $G \oplus H$ is the clique sum which identifies $\{v_1, \ldots, v_k\}$ with $\{w_1, w_2, \ldots, w_k\}$ where $k < p$. From Corollary 2.25, we know that the cop can protect $|V(G \oplus H)| - \max\{|V(G)|, |V(H)|\} + k$ vertices in $G \oplus H$. However,

given that the maximal cliques containing $\{v_1, v_2, \ldots, v_k\}$ and $\{w_1, w_2, \ldots, w_k\}$ have a larger order, we know there is some vertex v_j, where $k < j \leq p$, such that v_j is in the neighbourhood of the cop's position in $G \oplus H$ but is not protected by the cop, since v_j is not identified in the clique sum. Therefore, when $G \oplus H$ identifies non-maximal cliques, we know the cop can protect more vertices than outlined in Corollary 2.25 and therefore this is not an optimal strategy for the cop.

Both Theorem 2.23 and Corollary 2.25 have the cop remaining stationary for the entirety of the game, unless capture is possible on the identified clique. Sometimes this is the optimal strategy for the cop, as is the case for a path of order four. Suppose $V(P_4) = \{v_1, v_2, v_3, v_4\}$, since $P_4[\{v_2, v_3, v_4\}]$ forms a subgraph isomorphic to $K_{1,2}$ we can place a cop on v_3 and protect vertices v_2, v_3 and v_4. From Theorem 2.6, we know that $dmg(P_4) \geq 1$ since $rad(P_4) = 2$. Therefore the graph P_4 is tight to Theorem 2.23.

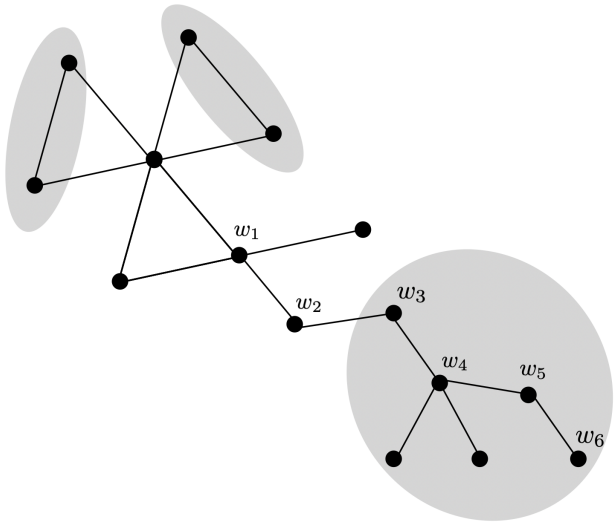


Figure 2.11: A graph H, with star cut-set $N[w_1]$ and its connected components.

However, having a cop remain stationary is not always optimal for the cop, especially in cases where the cop can still prevent the robber's access to the star cut-set while moving to different vertices in the graph. For example, consider a graph G with star cut-set S where $M(G) = \mu(S)$. If the largest component in $G[V(G) \setminus S]$ is a tree, the cop can chase the robber down the tree while still protecting vertices in S. See Figure 2.11, where the cop can protect $|V(H)| - 4$ vertices by starting on w_1 and moving down the path from w_1 to w_6. However, if we do not know the structure of the largest component in $G[V(G) \setminus S]$, we cannot necessarily guarantee that the cop

can move off their starting position without the robber being able to access vertices in S without being captured.

2.5 Cactus Graphs

We begin by providing a definition for blocks in a graph and define a family of graphs, cactus graphs, that restrict the structure of blocks. Following this, we provide an upper bound for the damage number of cactus graphs.

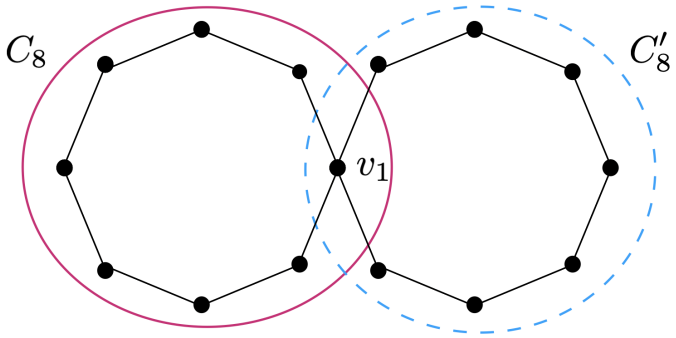


Figure 2.12: A graph $C_8 \oplus C'_8$ where any one vertex in C_8 is identified with any one vertex in C'_8. We label this identified vertex as v_1.

For a graph with a star cut-set, it is not always optimal for the cop to remain on one vertex for the entirety of the game. Consider the graph $C_8 \oplus C'_8$ in Figure 2.12, note that $N[v_1]$ forms a star cut-set in the graph. Given Theorem 2.23, we know that the cop can protect at least ten vertices in the graph by placing themselves on v_1 (every vertex in either C_8 or C'_8 and $N[v_1]$). However, if the cop was to move off the vertex v_1, instead of remaining stationary, they can protect more vertices. Without loss of generality, assume the robber places themselves on cycle C_8, we know that they must place themselves a distance of two away from the cop. The cop can then protect an additional two vertices using the strategy described by Theorem 1.8. The cop passes on the first round and moves in the opposite direction of the robber, unless the robber passes, in which case the cop does as well. Using this strategy the cop is able to restrict the robber's access to $N[v_1]$ for the entirety of the game. Therefore $dmg(C_8 \oplus C'_8) \leq 3$. Since we know the structure of the connected components in $C_8 \oplus C'_8$ we could move the cop off their initial vertex v_1 and protect more vertices in $C_8 \oplus C'_8$. The graphs C_8 and C'_8 form *blocks* in the graph $C_8 \oplus C'_8$.

Definition 2.26. *Suppose G is a graph with at least one cut-vertex. Define*

$$B(G) = \{B_1, B_2, \ldots, B_m\}$$

*to be the set of maximal induced subgraphs of G such that each B_i contains no cut-vertices, for $i \in \{1, 2, \ldots, m\}$. We say that each B_i is a **block** of G, and $B(G)$ is the **set of blocks**.*

Note that each $B_i \in B(G)$ will contain a vertex that is a cut-vertex in G but not in B_i. We can restrict the graph structure of each block to define the family of cactus graphs.

Definition 2.27. *A connected graph G is a **cactus graph**, if for all $B_i \in B(G)$, B_i is isomorphic to C_n for some $n \geq 2$.*

Figure 2.12 is a cactus graph since C_8 and C_8' are the only blocks in $C_8 \oplus C_8'$. We will now generalize a strategy for the cop to protect at least $M(G)$ vertices in a cactus graph, G. We present these results in multiple steps.

Lemma 2.28. *Let v be a cut-vertex in a cactus graph G. Suppose $v \in B_i$ where $B_i \in B(G)$ and $|V(B_i)| = m$. If the cop places themselves on vertex v initially, the cop can protect at least $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in B_i.*

Proof. Suppose the cop places themselves on vertex v initially. Note that since v is a cut-vertex in G, then it is a vertex in more than one block in $B(G)$. For this lemma, we are only concerned with the cop protecting vertices in one block. Therefore no matter how many blocks v is in, without loss of generality, we can consider just one block, B_i.

If B_i is isomorphic to C_m where $m \in \{2, 3, 4\}$, then the cop can clearly protect $m - \lfloor \frac{m-1}{2} \rfloor$ vertices by remaining on vertex v. Therefore assume that B_i is isomorphic to C_m where $m \geq 5$. We now need to consider the neighbourhoods of each vertex in C_m, to see how we can modify our strategy in Theorem 1.8 for B_i. Since every block in $B(G)$ is isomorphic to some C_n, we know that for any two cycles in G, they share at most one vertex. As a result, if the robber is on vertex $u \notin V(B_i)$ but $d(u, w) = 1$ for some $w \in V(B_i)$ then we know that $u \not\sim x$ for all $x \in V(B_i) \setminus \{w\}$. Therefore we define w as the shadow of u, since this is a vertex distance one away from the robber in B_i.

Suppose the robber does not enter B_i for the entirety of the game, then the cop can enact the strategy in Theorem 1.8 and protect at least $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in B_i. We can modify the strategy in Theorem 1.9 in the following way. Suppose the cop passes on the first round, if the robber decides to enter B_i then assume the robber is on vertex r_i on round i:

- If $r_i \in V(B_i)$, then the cop can use the same strategy in Theorem 1.8.
- If r_i is a distance of one away from B_i, then the cop can play the same strategy in Theorem 1.8 but play as if the robber is on the shadow of r_i.
- If r_i a distance of k away from B_i where $k \geq 2$. If $k < m$ then the cop plays the strategy in Theorem 1.5 as if the robber is a distance of k away in C_m. If $k \geq m$, then the cop can play the same strategy as if the robber was a distance of $m - 1$ in C_m.

Using this modified strategy, the cop can guarantee to protect at least $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in B_i. □

We now provide definitions of new graph parameters of cactus graphs that will help define an upper bound of the damage for these graphs.

Definition 2.29. *Let v be a cut-vertex in a cactus graph G. For each block $B_i \in B(G)$ where $v \in V(B_i)$ we define the* ***block cycle protection number*** *of a block B_i and vertex v as $m - (\lfloor \frac{m-1}{2} \rfloor)$ where $m = |V(B_i)|$. We denote it as $\rho_v(B_i)$.*

In Figure 2.12, the block cycle protection number of blocks C_8 and C'_8 and vertex v_1 are $\rho_{v_1}(C_8) = \rho_{v_1}(C'_8) = 5$.

Definition 2.30. *Let G be a cactus graph. For a cut-vertex $v \in V(G)$ we say the* ***cactus cut component number*** *of a cut-vertex v is the maximum number of $|V(G_j)| - \rho_v(B_i) + 1$ for all connected components, G_j in $G[V(G) \setminus \{v\}]$ and B_i is the unique block $B_i \in B(G)$ where $v \in V(B_i)$ and $B_i[V(B_i) \setminus \{v\}] \subseteq G_j$. We denote the cactus cut component number of v as $P_v(G)$.*

In Figure 2.12, $P_{v_1}(C_8 \oplus C'_8) = 3$ since the two connected components in $G[V(C_8 \oplus C'_8) \setminus \{v_1\}]$ are both paths of order seven, we need to only consider one connected component, P_7. Note that $v_1 \notin V(P_7)$ but $v_1 \in V(C_8)$. Therefore $C_8[V(C_8) \setminus \{v_1\}] = P_7$. So then we have that $|V(P_7)| - \rho_{v_1}(C_8) + 1 = 7 - \big(8 - (\lfloor \frac{8-1}{2} \rfloor)\big) + 1 = 7 - (8-3) + 1 = 3$.

Theorem 2.31. *Suppose G is a cactus graph where v is the vertex where eccentricity of v is equal to $rad(G)$:*

- *If v is a cut-vertex then $dmg(G) \leq P_v(G)$.*
- *If not, consider $v_1, \ldots, v_n$ where for each v_i, v_i is a cut-vertex in $V(B_i)$, where $B_i \in B(G)$ and $v \in B_i$. Let v_j be the vertex such that $P_{v_i}(G)$ is minimized for all $i \in \{1, \ldots, n\}$. Then $dmg(G) \leq P_{v_j}(G)$.*

Proof. To begin, we will assume that v is a cut-vertex. Suppose we place a cop on vertex v. Since v is a cut-vertex, we know that once the robber places themselves on some connected component, G_j of $G[V(G) \setminus \{v\}]$ the robber will have to access vertex v to move to a different component. Therefore, if the cop is able to protect vertex v for the entirety of the game, then the robber cannot move to any other connected component in $\{G_k|\ G_k \in G[V(G) \setminus \{v\}],\ k \neq j\}$. In such a case, the cop can protect all vertices in $\{G_k|\ G_k \in G[V(G) \setminus \{v\}],\ k \neq j\}$.

From Lemma 2.28, we know that the cop can protect $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in B_i where $B_i \in B(G)$, $v \in V(B_i)$ and $|V(B_i)| = m$. Notice in the proof for Lemma 2.28, the robber is never able to damage vertex v because every vertex that the cop occupies throughout, is protected in B_i. Since the cop beings on vertex v initially, we can guarantee the cop will occupy vertex v, no matter the starting position of the robber. With this, we can conclude that the cop can protect an additional $m - \lfloor \frac{m-1}{2} \rfloor - 1 = \rho_v(B_i)$ vertices in the connected component G_j, where G_j is the connected component in $G[V(G) \setminus \{v\}]$ that the robber places themselves in initially.

Considering the robber wants to damage the maximum number of vertices in the graph, they will want to place themselves in the component such that the number $|V(G_j)| - \rho_v(B_i) + 1$ is minimized. This the number of vertices that the robber can damage in the component G_j. Therefore we know that $dmg(G) \leq P_v(G)$ since $P_v(G) = |V(G_j)| - \rho_v(B_i) + 1$.

Now suppose that v is not a cut-vertex. We do not want to place the cop on vertex v initially because $G[V(G) \setminus \{v\}]$ does not result in disjoint components. Therefore we will want to consider cut-vertices in the same block as v and determine which vertex minimizes the cactus cut component number. Let $v_1, v_2, \ldots, v_n$ be the vertices such that for each v_i, v_i is a cut-vertex in $V(B_i)$, where $B_i \in B(G)$ and $v \in B_i$. Lemma 2.28, we know that the cop can protect $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in B_i, where $v \in V(B_i)$ and $|V(B_i)| = m$.

Assume the cop places themselves on some vertex v_i where $i \in \{1, 2, \ldots, n\}$.

Again, since v_i is a cut-vertex, we know that once the robber places themselves on some connected component, G_t of $G[V(G) \setminus \{v_i\}]$ the robber will have to access vertex v_i to move to a different component. Therefore, if the cop is able to protect vertex v_i for the entirety of the game, then the robber cannot move to any other connected component in $\{G_k|\ G_k \in G[V(G) \setminus \{v_i\}],\ k \neq t\}$. From Lemma 2.28, we know that the cop can protect $m - \lfloor \frac{m-1}{2} \rfloor$ vertices in some $B_p \in B(G)$ where $v_i \in V(B_p)$ and $|V(B_p)| = m$, one of which is vertex v_i. With this, we can conclude that the cop can protect an additional $m - \lfloor \frac{m-1}{2} \rfloor - 1 = \rho_v(B_p)$ vertices in the connected component G_t where G_t is the connected component in $G[V(G) \setminus \{v_i\}]$ that the robber places themselves in initially.

Again, considering the robber wants to damage the maximum number of vertices in the graph, they will want to place themselves in the component such that the number $P_{v_i}(G) = |V(G_t)| - \rho_v(B_p) + 1$ is minimized since this is the number of vertices the robber can damage given the strategy of the cop. Let v_j be the the vertex such that $P_{v_i}(G)$ is minimized for all $i \in \{1, 2, \ldots, n\}$. Therefore we know that $dmg(G) \leq P_{v_j}(G)$ □

In the proof of Theorem 2.31, we did not want to initially place the cop on the center of the graph if it was not a cut-vertex. Consider the graph H in Figure 2.13. Here v is the center of H, however it is not a cut vertex in H. Therefore, instead of placing themselves on v, the cop will want to asses the number of vertices the robber can damage if they place themselves on vertex v_1, v_2 or v_3 and employ the strategy in the proof of Theorem 2.31. Notice that $H[V(H) \setminus \{v_3\}]$ results in two connected components: an isolated vertex u, where v_3 is its unique neighbourhood, and $H[V(H) \setminus \{v_3, u\}]$. Therefore, if the robber was to place themselves on vertex v_3 and use the above strategy, then $P_{v_3}(H) = 13$, which would mean the cop can only protect at most seven vertices using this strategy. However, notice that $H[V(H) \setminus \{v_1\}]$ results in two connected components. Let $S = \{x|\ x \in V(H), x \in N(v_1), x \in B_i\}$ and $S^c = \{x|\ x \in V(H), x \in N(v_1), x \notin B_i\}$ where $B_i \in B(G)$ and B_i contains v. Then $H[V(H) \setminus \{v_1\}]$ has connected components, $H[V(H) \setminus S]$ and $H[V(H) \setminus S^c]$. Therefore $P_{v_1}(G) = 10$, which is smaller than $P_{v_3}(G)$. Furthermore $H[V(H) \setminus \{v_1\}] \cong H[V(H) \setminus \{v_2\}]$, so $P_{v_2}(G) = 10$ as well. By assessing the cactus cut component number of each cut-vertex in the block of v, we can ensure the cop can protect the most vertices in H using the same strategy described in Theorem 2.31.

There are some cases where Theorem 2.31 and Theorem 2.23 provide the same

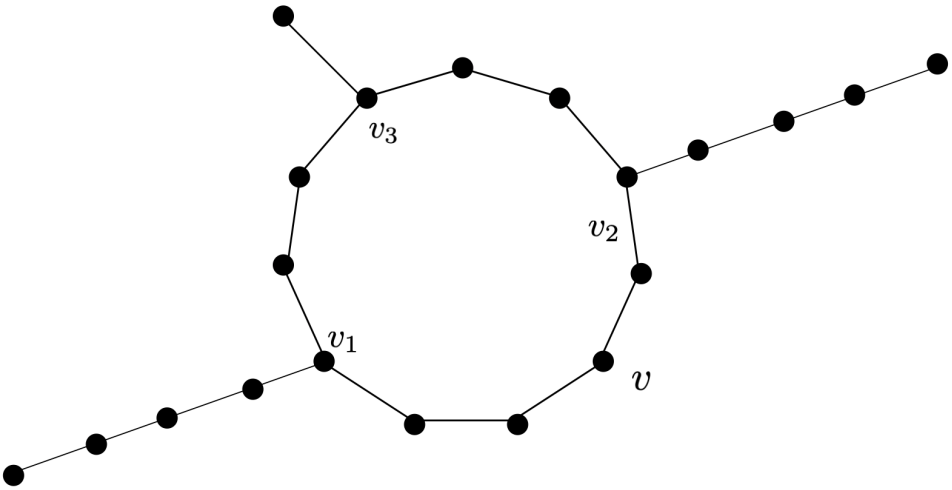


Figure 2.13: A cactus graph H, where v is center of H and $\{v_1, v_2, v_3\}$ is the set of the block C_{11}.

upper bound for a cactus graph. Suppose that G_j is the component the robber places themselves in a cactus graph where $|V(G_j)| - \rho_{v_t}(B_i) + 1$ is minimized for the unique block B_i where $v_t \in B_t$ and $B_i[V(B_i) \setminus \{v_t\}] \subseteq G_j$ and v_t is either the center of the graph or in the same block as the center of the graph. If $B_i \cong C_m$, where $m \in \{2, 3, 4\}$ then we know that $\rho_{v_t}(B_i)$ vertices can be protected by remaining on the vertex v_t. In such cases, $P_{v_t}(G) = M(G)$, since the star cut-set induced by $N[v_t]$ will minimize the star cut area when v_t is the center of the graph or a star cut-set in B_i. Note P_4 is a cactus graph where the center of the graph is a cut-vertex, v_t. We then know that $B_i \cong C_2$ for all $B_i \in B(P_4)$, so $P_{v_t}(P_4) = M(P_4) = 1$. Since $dmg(P_4) = M(P_4)$, as shown in Section 2.4, then we know that $dmg(P_4) = P_{v_t}(P_4)$. Consequently, we know that Theorem 2.31 is tight for some graphs.

Chapter 3

Future Work

This section will discuss specific questions to guide future research that is related to earlier discussion of both work in [8] and new results provided in Chapter 2. Section 2.1 began with a characterization of graphs with damage number zero. Following this, we characterized graphs with damage number one. Can a characterization be found for graphs with a larger damage number?

Problem 3.1. *For a graph G with order n and an integer i where $2 \leq i \leq n$, what conditions on G imply $dmg(G) = i$?*

Problem 3.1 becomes increasingly harder with every increment of i. For example, when considering graphs with damage number of one, we knew the positioning of the robber for the entire game. As Lemma 2.3 indicates, if $dmg(G) = 1$ there is no benefit for the robber to move during the game, therefore the starting position of the robber will be their position for the entire game. The same cannot be said for graphs with damage number two or greater, since the robber needs to damage more than one vertex, they must move off their starting position. Therefore we can guarantee the robber's location is changing throughout the game, which complicates our analysis.

In Section 2.2, we presented an improved lower bound in Theorem 2.6, $dmg(G) \geq rad(G) - 1$. This bound is stronger than $dmg(G) \geq \lfloor \frac{rad(G)}{2} \rfloor$, which is the lower bound in [8]. We also briefly discussed classes of graphs that are tight to this bound. These included infinite classes, $P_2 \square P_m$, $P_3 \square P_m$, $P_2 \boxtimes P_m$, and $P_3 \boxtimes P_m$ for any $m \in \mathbb{N}$. Can we generalize the strategy for the cop in $P_3 \square P_m$ and $P_3 \boxtimes P_n$ for all possible path lengths?

Problem 3.2. *For paths P_n and P_m for order n and m respectively, for some $n, m \geq 4$, is $dmg(P_n \square P_m) = rad(P_n \square P_m) - 1$?*

Problem 3.3. *For paths P_n and P_m for order n and m respectively, for some $n, m \in \geq 4$, is $dmg(P_n \boxtimes P_m) = rad(P_n \boxtimes P_m) - 1$?*

In fact, we can further generalize this question and ask what property of a graph guarantees that it is tight to Theorem 2.6.

Problem 3.4. *What conditions on a graph G ensure $dmg(G) = rad(G) - 1$?*

In Section 2.4, we explored the relationship between cut-sets and the damage number of a graph, by defining a new property, star cut-sets. As indicated in the discussion of Theorem 2.23, the cop remains stationary to ensure the robber can only damage at most $M(G)$ vertices in the graph G. First, it would be beneficial to characterize graphs where the cop's stationary strategy is optimal. Once a characterization exists, we can disregard these graphs when analyzing how to mobilize the cop in the strategy for Theorem 2.23 to protect more vertices in a graph.

Problem 3.5. *What graphs G with star cut-sets, have $dmg(G) = M(G)$?*

We know that such graphs exist through our discussion of P_4 in Section 2.4. For a graph G where $dmg(G) < M(G)$, can we develop a general strategy for the cop to protect more vertices in G? Suppose G has star cut-set S where $M(G) = \mu(S)$. Again, as we described in our discussion of Theorem 2.23, without knowing the structure of the largest component in $G[V(G) \setminus S]$ it is difficult to determine if the cop can move off their starting position without the robber being able to access vertices in S and not be caught immediately.

Problem 3.6. *For a graph G of order n with a star cut-set S, if $dmg(G) \neq M(G)$ can the cop move off their starting position to protect more than $n - M(G)$ vertices in G?*

The solution to this problem would help improve the upper bound for graphs with star cut-sets. Furthermore, it might help develop a strategy for the cop to move from the star cut-set S to a different star cut-set in the component $G_i \in G[V(G) \setminus S]$ where G_i is the component the robber places themselves in initially. By doing so, the cop might be able to protect more vertices in G_i by restricting the robber's access to a proper subset of vertices in G_i.

We now discuss open problems that are closely connected to the work done by Cox and Sanaei. Recall that Theorem 1.9 presented a general upper bound for any graph G with n vertices: $dmg(G) \leq n - \Delta(G) - 1$.

Problem 3.7 ([8])**.** *What families of graphs have an extremal damage number, that is* $dmg(G) = n - \Delta(G) - 1$*?*

In Section 1.5.4, we investigated Cox and Sanaei's results on maximum damage numbers for graphs with eight vertices or less. We could expand further to investigate extreme damage numbers of graphs with a larger number of vertices.

Problem 3.8. *For any graph* G *of order* n *where* $n \geq 9$*, what is the maximum damage number of* G*?*

In [8], Cox and Sanaei identified only one graph in which $dmg(G) > \left\lfloor \frac{n}{2} \right\rfloor$, which was helpful in determining that the robber has the ability to damage more than half the vertices in a graph. This leads us to the following question.

Problem 3.9. *What properties must* G *contain in order for* $dmg(G) > \left\lfloor \frac{n}{2} \right\rfloor$*?*

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